Bayesian inference for inverse obstacle scattering problems

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Outline

1. Introduction
2. Forward model
3. Bayesian inference
4. MCMC techniques
5. Simulation study
6. Conclusion and issues

Introduction

Basic scattering problem :
Concerns the effect of an inhomogeneous medium on an incident acoustic wave.

$$u \text{ (total field)} = u_i \text{ (incident field)} + u_s \text{ (scattered field)}$$

Inverse problem :
Determine the shape or physical properties of the obstacle from the measurement of u_s.

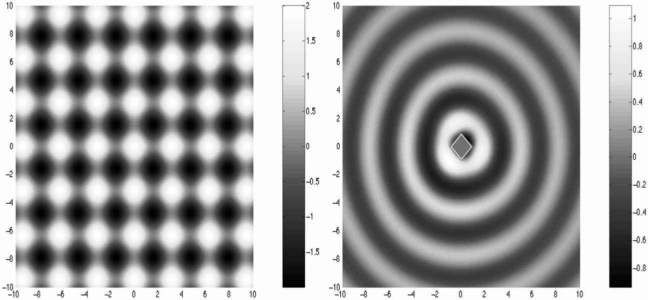

Figure: The real part of u_i (Left) and u_s (Right) for a diamond shaped obstacle

Direct Scattering Problem

Determine u_s from knowledge of u_i and the scattering obstacle.
Assuming the time harmonic waves

$$U(x,t) = u(x)e^{-iwt}$$

the u_s is estimated using the following equations,

$$\begin{aligned} \text{Helmholtz equation} &: \quad \nabla^2 u + k^2 u = 0 \ \text{ in } \Omega \in \mathbb{R}^2 \\ \text{Neumann B.C} &: \quad \frac{\partial u}{\partial n} = 0 \ \text{ on } \partial\Omega \\ \text{Radiation condition} &: \quad \lim_{r\to\infty} r\left(\frac{\partial u_s}{\partial r} - iku_s\right) = 0 \end{aligned}$$

where
k : wave number
n : the unit outward normal to $\partial\Omega$
$r = |x|$.

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The Neumann boundary condition corresponds to a sound-hard obstacle and the Sommerfield radiation condition guarantees that u_s is outgoing.

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\text{Radiation condition} \quad &: \quad \lim_{r\to\infty} r\left(\frac{\partial u_s}{\partial r} - iku_s\right) = 0
\end{aligned}$$

- Also called the *forward map*, $\mathcal{M}_f$ since it maps the image of the obstacle to the scattering field (i.e., $\mathcal{M}_f$:Image$\rightarrow u_s$).

Direct Scattering Problem

- Solving a forward map for u_s becomes a exterior boundary value problem (BVP)
- Numerical and analytical solutions for the exterior Helmholtz equation.
- Analytical solution for a limited case.
- Numerical solution using the Green's function and Green's formulae

Exterior boundary value problem (BVP)

For a sufficiently smooth $\partial\Omega$, the solution for u_s is

$$\int_{\partial\Omega}\left[\frac{\partial u_i(x)}{\partial n}g(x|\xi)+\frac{\partial g(x|\xi)}{\partial n}u_s(x)\right]dl(x) \quad = \quad \begin{cases} u_s(\xi)\ ,\ \xi\in\Omega \\ \dfrac{u_s(\xi)}{2}\ ,\ \xi\in\partial\Omega \end{cases}$$

where

$$g(x|\xi)=\frac{i}{4}H_0(k|x-\xi|)\ , \qquad \frac{\partial g(x|\xi)}{\partial n}=-\frac{ik}{4}H_1(k|x-\xi|)\frac{\partial(|x-\xi|)}{\partial n}\ ,$$

H_0 : a Hankel function of the first kind of order zero
H_1 : a Hankel function of the first kind of order one

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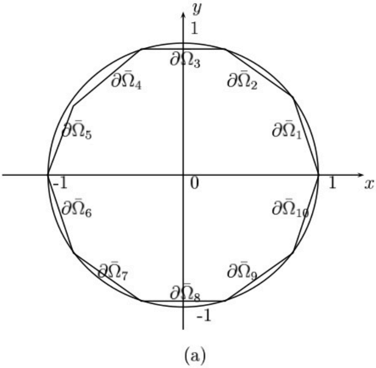
This is solved numerically using the boundary element method (BEM).
Boundary $\partial\Omega$ is discretized by N_b number of elements, $\{\partial\bar{\Omega}_i\}_{i=1}^{N_b}$
i.e., $\partial\Omega \approx \cup_{i=1}^{N_b}\partial\bar{\Omega}_i$

Figure: (a) the set of boundary elements $\{\partial\bar{\Omega}_i\}_{i=1}^{10}$ and the original boundary which is a unit circle and (b) the measurement points $s_1, \ldots, s_{10}$ around an unit circle obstacle.



Inverse Scattering Problem

- Unknown shape of the obstacle from the measurement of the scattering field u_s
- Nonlinear and ill-posed problem
- Small variations in u_s can lead to large errors in the reconstruction of the obstacle

Statistical Inference

Using a Bayesian approach, the inverse problem is tackled as a statistical inference for an unknown shape of the obstacle

- Γ - the continuous state space of feasible images w
- d - noisy measurements of the far field patterns

The posterior density for w is

$$p(w|d) = \frac{p(d|w)p(w)}{\int_\Gamma p(d|w)p(w)dw}$$

Statistical Inference

- *Synthetic Data* - u_s (synthetic noise free data) respect to the true image is obtained by solving the *forward map*. We assume both the real and imaginary parts of each measurement contain zero-mean Gaussian-distributed noise in reality.
- *Prior*,($p(w)$) - Since there is no real prior knowledge of w and hence no subjective prior. A uniform distribution over Γ is used.
- *Likelihood*, $p(d|w)$

$$p(d|w) = \prod_{i=1}^{N_f} \exp\left(\frac{-(\mathrm{Re}(d(s_i) - u_s(s_i)))^2 - (\mathrm{Im}(d(s_i) - u_s(s_i)))^2}{2\sigma^2}\right)$$

N_f : the number of measurements

Statistical Inference

The expected shape of obstacle is

$$\mathbb{E}[w] = \int_\Gamma wp(w|d)dw \ .$$

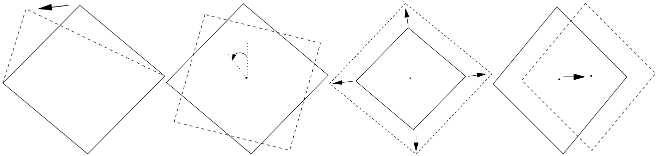
If the sample set $\{w_i\}_{i=1}^N$ generated from $p(w|d)$ over Γ the expectation of w is estimated by Monte Carlo integration

$$\bar{w} \approx \frac{1}{N} \sum_i w_i \ .$$

Markov chain Monte Carlo method is used to generated samples from $p(w|d)$, and the central limit theorem (CLT) holds for $\bar{w}$.

Markov chain Monte Carlo (MCMC) method

Figure: Four types of proposal movements; $\mathcal{V}$, $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ (Left to Right)



MH-Algorithm

Given a state $W_n = w$, the Metropolis Hastings algorithm is as follows

Step 1 Select a move m_i. $i \in \{1, \dots, N_m\}$ with probability ε_i. Generate w' by sampling $\Phi_i(w \to w')$.

Step 2 Compute the acceptance probability for the state, $\alpha^{m_i}(w \to w')$.

Step 3 Accept w' with $\alpha^{m_i}(w \to w')$.

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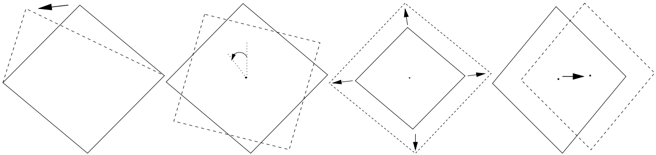


Move $\mathcal{R}$ Rotate w by a random angle $h^{\mathcal{R}}$ with respect to the center of mass, $h^{\mathcal{R}} \sim U(-\delta^{\mathcal{R}}, \delta^{\mathcal{R}})$

Move $\mathcal{T}$ Shift w by a random vector $h^{\mathcal{T}}$, $h^{\mathcal{T}} \sim U(-\delta^{\mathcal{T}}, \delta^{\mathcal{T}})^2$

Move $\mathcal{V}$ Move a position of one vertex by a random vector $h^{\mathcal{V}}$, $h^{\mathcal{V}} \sim U(-\delta^{\mathcal{V}}, \delta^{\mathcal{V}})^2$

Move $\mathcal{S}$ Change a size of w by a random rate $h^{\mathcal{S}}$ respect to the center of mass, $h^{\mathcal{S}} \sim U(\frac{1}{\delta^{\mathcal{S}}}, \delta^{\mathcal{S}})$

Computing the Likelihood efficiently

- Precomputation of Hankel function : Instead of evaluating a truncated infinite sum or similar for each Greens function value, look up the table of precomputed Hankel function values.
- Efficient boundary discretization : In solving the forward problem numerically, the number of elements N_b relates to the size of dimensions of the linear system.

Delayed acceptance MCMC (DAMCMC)

- Generates samples from the exact posterior using an intermediate approximation step.
- If a proposal is accepted by the approximation, it is corrected by calculating the true posterior density to ensure it reaches the target distribution. Otherwise a proposal is rejected.
- Computation time reduction by avoiding calculation of the exact density for proposals that are rejected.
- Quality of approximation.
- The speed up of these algorithms over the standard MH algorithm equals the inverse of the proportion of acceptance.

Delayed acceptance MCMC (DAMCMC)

Algorithm

Let $W_n = w$ and $p^*_w(w'|d)$ denotes the approximation to $p(w'|d)$ computed at w (i.e., $p^*_w(w|d) = p(w|d)$).

Step 1 Select a move m_i. $i \in \{1, \ldots, N_m\}$ with probability ε_i. Generate w' by sampling $\Phi_i(w \to w')$.

Step 2 Using the approximation of the present section, estimate $p^*_w(w'|d)$.

Step 3 Compute the Metropolis Hastings ratio $\alpha^{m_i}_{MH}(w \to w')$ using $p^*_w(w'|d)$.

Step 4 Accept or reject the candidate state w':

4.1 If w' is rejected, set $W_{n+1} = w$ and go back to **Step 1**.

4.2 If w' is accepted, compute $p_w(w'|d)$ using the exact calculation. Compute the second-stage acceptance probability $\alpha^{m_i}(w \to w')$ and accept w' with $\alpha^{m_i}(w \to w')$.

Forward map approximation

- Linear approximate
  Use of the the Fréchet derivative of the forward map.
- Coarse discretization approximate
  Use of a coarsened boundary discretization using fewer boundary elements than the exact calculation.

Forward map approximation

- Linear approximate (Hettlich (1995))
  For sufficiently small vector fields $h \in C^1(\partial\Omega)$ a perturbation of $\partial\Omega$ is a boundary of a domain denoted by $\partial\Omega_h = \{x' \in \mathbb{R}^2 : x' = x + h(x), x \in \partial\Omega\}$ in the class C^1. The normal component of a vector field h is denoted by $h_n = h \cdot n$ and the notation $(\nabla u)_t$ for the tangential component $(\nabla u)_t = n \times (\nabla u \times n)$ of a vector field u is used. Then the operator $\mathcal{M}$ is Fréchet differentiable at $\partial\Omega$, i.e.,

$$\frac{1}{\|h\|_{C^1(\partial\Omega)}} \|\mathcal{M}(\partial\Omega_h) - \mathcal{M}(\partial\Omega) - \mathcal{M}'(\partial\Omega)h\| \to 0 \ , \qquad h \to 0 \ .$$

  Note that the derivative is related to u'_∞, (i.e., $\mathcal{M}'(\partial\Omega)h = u'_\infty$). u'_∞ is the far-field pattern of the radiating solution of

$$\nabla^2 u'_\infty + k^2 u'_\infty = 0 \qquad \text{in } \Omega \subset \mathbb{R}^2$$

$$\frac{\partial u'_\infty}{\partial n} = k^2 h_n u_s + \mathrm{Div}(h_n (\nabla u_s)_t) \qquad \text{on } \partial\Omega \ .$$

  u_s denotes the solution of the scattering problem with respect to Ω. The linear approximation u'_s to the scattering field on a domain Ω_h for small h is given by

$$u'_s \simeq u_s + u'_\infty \ .$$

- Coarse discretization approximate

Forward map approximation

- Linear approximate
- Coarse discretization approximate
  - Fine discretization gives an accurate solution and demands a higher computation cost..
  - Coarse discretization using fewer boundary elements but also a reasonably accurate solution.
  - Separation analysis of coupled Markov chains (Nicholls, G. K., Fox, C. and Watt, A. (2012))
  - Key idea is to compare the mean separation time of two chains under the same random sequences and if the time is long enough, two chains are empirically identical.

- Linear approximate
- Coarse discretization approximate

  Algorithm

  Generate W_0 form the initial state distribution ρ_0 and set $Z_0 = W_0$.

  For $n < N_t$,

  Step 1 Generate a proposal w' from $\psi(w'|w)$.

  Step 2 Compute the acceptance probabilities; $\alpha_n^w(w'|w)$ using $p_w(w'|d)$ and $\alpha_n^z(w'|w)$ using $p_z(w'|d)$.

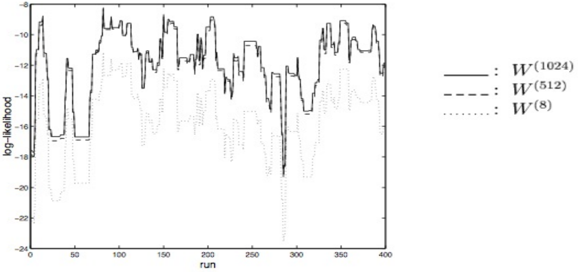
  Step 3 Accept w' with $\alpha_n^w(w'|w)$. If w' is accepted set $W_{n+1} = Z_{n+1} = w'$. Otherwise set $W_{n+1} = Z_{n+1} = w$.

  Step 4 After N_t iterations compute the mean absolute differences of α_w and α_z.

$$\alpha_{w,z} = \frac{1}{N_t} \sum_{n=1}^{N_t} |\alpha_n^w - \alpha_n^z|$$

- Linear approximate
- Coarse discretization approximate

Figure: log of likelihood of $W^{(1024)}$, $W^{(512)}$ and $W^{(8)}$

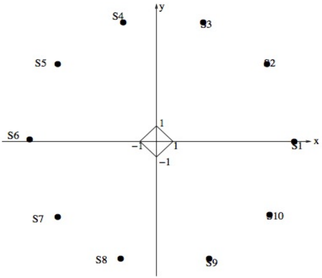


- Linear approximate
- Coarse discretization approximate
  - For a large N_t, $\alpha_{w,z}$ is the average separation probability per update and the inverse relates to the mean separation time of Z from W.
  - For theoretical result, see Nicholls, G. K., Fox, C. and Watt, A. M. (2012).
  - If a separation time is longer than the total run length, the approximate chain is identical to the exact chain.
  - If $\alpha_{w,z}$ is greater than an autocorrelation time of the exact, we treat that the chain mixes faster than it separates.

Simulation study

- A diamond shape of obstacle.
- Ten u_s measurements points, $(10\cos(\phi_i), 10\sin(\phi_i))$ where $\phi_i = 2\pi(i-1)/10$, $i = 1, \ldots, 10$.
- Two incident fields, $u_i^1 = e^{ikx}$ and $u_i^2 = e^{iky}$.
- Synthetic data : $d_i = u_s(s_i) + \epsilon_1 + i\epsilon_2$, $\epsilon_1, \epsilon_2 \sim N(0,\sigma^2)$.
- 512 boundary elements are used, $N_b = 512$.

Figure: A diamond-shaped obstacle and the measurement points $s_1, \ldots, s_{10}$



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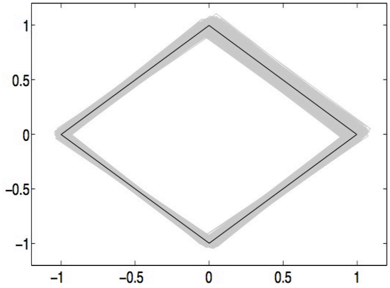
- Synthetic data : $d_i = u_s(s_i) + \epsilon_1 + i\epsilon_2$, $\epsilon_1, \epsilon_2 \sim N(0, \sigma^2)$.
- 512 boundary elements, $N_b = 512$, for the exact chain.

Figure: Empirical posterior mean (black line) and 95% confidence intervals (grey lines).



Simulation study

$[\varepsilon_{\mathcal{R}}\ \varepsilon_{\mathcal{T}}\ \varepsilon_{\mathcal{V}}\ \varepsilon_{\mathcal{S}}]$	chain	L	τ	$\tau \times t$	N
[0 0 1 0]	W	-10.7925±0.0218	71.9718	24.3731	600000
[0 0 1 0]	W^C	-10.7914±0.0246	90.1811	17.6190	600000
[0 0 1 0]	W^L	-10.7932±0.0457	312.4799	94.7306	600000
[0.07 0.07 0.80 0.06]	W	-10.7894±0.0227	79.1486	21.0694	600000
[0.07 0.07 0.80 0.06]	W^C	-10.8025±0.0255	97.8979	15.7486	600000
[0.07 0.07 0.80 0.06]	W^L	-10.8143±0.0486	344.6875	101.2250	600000
[0.10 0.10 0.70 0.10]	W	-10.7964±0.0233	82.2308	21.7089	600000
[0.10 0.10 0.70 0.10]	W^C	-10.7690±0.0260	102.0964	15.2867	600000
[0.10 0.10 0.70 0.10]	W^L	-10.8098±0.0438	294.3497	77.254	600000
[0.20 0.20 0.55 0.05]	W	-10.7639±0.0268	108.5473	28.8844	600000
[0.20 0.20 0.55 0.05]	W^C	-10.7696±0.0283	121.4257	21.7538	600000
[0.20 0.20 0.55 0.05]	W^L	-10.7625±0.0450	313.8210	82.5361	600000

Summary

- The inverse obstacle scattering problem using Bayesian inference
- Two attempts to increase the forward map calculation efficiency ; precomputation of Hankel function values and efficient boundary discretization.
- The delayed acceptance MH algorithm can be effective in reducing a computational workload when an appropriate approximate is used.

Future works

- Advanced MCMC techniques such as adaptive delayed acceptance M-H algorithm by Cui, T., Fox, C., and O'Sullivan, M. J. (2011).
- Alternative representation for an obstacle shape.
- Improvement of forward map approximate.

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Thank you for your attention!