WAVE PROPAGATION ACROSS A CRACK IN AN ICE SHEET

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ABSTRACT

This article presents a mathematical model and a method of solution for the wave propagation across a crack in an ice sheet. Wiener-Hopf (W-H) technique is used to derive analytical solutions of the problem, which leads to simple description of the conditions at the crack. The conditions at the crack are represented by a 4-by-4 matrix and a vector in the W-H solutions. Thus it is possible to study the effects of the crack on the wave propagation using the matrix and the vector. As a beginning of mathematical modelling of the crack conditions, the connection at the crack is mimicked by a rotational and a vertical springs. These springs transmit the vertical and gradient differences from one ice sheet to the other. The wave propagation is studied under varying spring constants.

INTRODUCTION

This paper presents a W-H solution of wave propagation across a crack where two semi-infinite floating elastic plates are joined by various transition conditions (see Fig. 1). There has been a steady development of analytical solutions of floating elastic plates and water waves. A simple case of water-plate interaction problem is solved in (Balmforth and Craster, 1999) and (Evans and Davies, 1968) using the W-H technique. (There are too many papers on the W-H to list them all here.) A similar technique is used in (Chung and Fox, 2002) to solve a plate-plate interaction with an open gap and a rigid joint. In a slightly different situation, (Chung and Linton, 2003) gives analytical solutions of plate-water-plate problem using the residue calculus technique. A complementary problem, water-plate-water with free edge conditions is solved using the W-H in (Tkacheva, 2002). Wave scattering by a long crack is studied in

(Evans and Porter, 2003) using a Green's function for a floating elastic plate. The family of problems concerning floating elastic plates, which can be solved analytically, is growing. However, the conditions at the edges of the plates have been usually assumed to be free. In this paper we exploit the fact that the W-H technique can, with some modifications shown in (Chung and Fox, 2002), incorporate more complex transition conditions than the simple open gap or rigid joint.

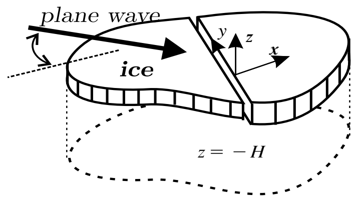

Figure 1: Schematic drawing of the crack in an ice sheet.

Our motivation here is to model the cracks in sea-ice sheets, because wave propagation into the ice field plays an important role when the ice sheets around the coast of Antarctica are broken up every year. We use a theoretical model of the ice sheet that assumes ice thickness, mass density and Young's modulus to be constant. This model is often used to study the dynamics of ice sheets with fairly homogeneous appearance. We focus on a relatively long and straight crack in an ice sheet. Such a crack can have a variety of physical properties depending on how it is formed, for example partially frozen slosh or solidly re-frozen crack. However, the mechanism of wave propagation across such a crack is not yet well understood.

In order to model varying characteristics of the crack, we introduce theoretical transition conditions using two springs linking the ice sheets (see Fig. 2). The vertical spring transmits the shear force that is determined by the displacement difference. The rotational spring transmits the bending moment that is determined by the difference of the gradient at the edges of the ice sheets.

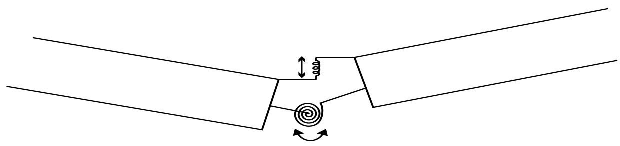

Figure 2: The rotational and the vertical springs connecting the two ice sheets.

The solutions here are given as an expansion over the modes that exist in ice sheets. The coefficients of the expansion are linear combination of four constants that must be determined by the transition conditions. Although, the W-H technique has been used by several researchers, its strength, which is the simple use of the transition conditions, has not been exploited. From the transition conditions a system of equations for the constants can be formulated with a 4×4 matrix and a vector. We therefore focus on the formulation of the matrix and the vector to study the qualitative behaviour of the solutions.

METHOD OF SOLUTION

Governing equations

We here use thin plate (Kirchhoff plate) model for the ice sheets. The flexural rigidity of the ice sheets in $x < 0$ and $x > 0$ are denoted by D_1 and D_2, respectively. The flexural rigidity is calculated using $D_i = Eh_i^3/12\left(1-\nu^2\right)$, where h_i, $i = 1, 2$ are the thickness of the plates in $x < 0$ and $x > 0$ respectively. ν is Poisson's ratio and E is effective Young's modulus that is constant in the plate. A plane wave of radial frequency ω is coming from $x = -\infty$ at an angle θ. We assume that amplitude and frequency of the incident wave are small and low enough so that the water can be assumed to be incompressible and irrotational. Then, vertical displacement of the plates, $w(x, y, t)$, and velocity potential of water, $\phi(x, y, z, t)$ satisfy following partial differential equations (see (Evans and Davies, 1968) and (Balmforth and Craster, 1999)),

$$
\left.\begin{aligned}
p &= D_1 \nabla^4_{x,y} w + m_1 w_{tt}, \text{ for } x < 0,\\
p &= D_2 \nabla^4_{x,y} w + m_2 w_{tt}, \text{ for } x > 0,\\
w_t &= \phi_z,\\
\rho\phi_t + \rho g w + p &= 0,
\end{aligned}\right\} \text{at } z = 0 \tag{1}
$$
$$
\phi_z = 0 \quad \text{at } z = -H,
$$
$$
\nabla^2_{x,y,z}\phi = 0 \text{ in the water.}
$$

g, ρ and p are acceleration due to the gravity, mass density of sea water and pressure acted on the surface of the water, respectively. The mass density of each plate is denoted by $m_i = \rho_{\mathrm{i}} h_i$ (ρ_{i} is the mass density of the ice), $i = 1, 2$, respectively. The incoming plane wave obliquely incident from infinite can be written as $\mathrm{i}\, I \exp \mathrm{i}\left(\lambda x + \omega t\right)$, where I is amplitude of the wave (i is there to simplify the calculations later) and the wave number λ is determined by the incident angle of the plane wave to the x-axis. Since the system of the equations given in Eq. 1 are linear with respect to $\phi(x, y, z, t)$, we may express the solution as

$$
\phi(x, y, z, t) = \mathrm{Re}\left[\phi(x, z, \omega)\, e^{\mathrm{i}(ky+\omega t)}\right]
$$

where $\phi(x, z, \omega)$ (or $\phi(x, z)$ for simplicity) is the complex function of amplitude of the solution and k denotes the wave number in the y-axis, i.e., $k = \lambda' \sin\theta$ $\lambda' = \sqrt{\lambda^2 + k^2}$.

We scale (or non-dimensionalize) the system of equations given above using characteristic length, characteristic time denoted by l_i and t_i, and the ratio of each characteristic values, which are defined as

$$l_i = \left(\frac{D_i}{\rho g}\right)^{1/4}, \quad t_i = \sqrt{\frac{l_i}{g}}, \quad l_{\mathrm{r}} = \frac{l_1}{l_2}.$$

We here assume that $l_{\mathrm{r}} < 1$, i.e., the plate on the right is more rigid than the other. l_{r} may be zero when $x < 0$ is free-surface. If we denote the non-dimensional variables with the bar, then the non-dimensionalized variables of space and time using l_2 are $(\bar{x}, \bar{y}, \bar{z}) = (x, y, z)/l_2$ and $\bar{t} = t/t_2$. We omit the bar to avoid the clutter from now. Then, the system of equations given in Eq. 1 scaled by l_2 for $x < 0$ and $x > 0$ become

$$(2) \quad \begin{aligned} &\left\{l_{\mathrm{r}}^4 \left(\partial_x^2 - k^2\right)^2 - m_1\omega^2 + 1\right\} \phi_z(x,0) = \omega^2 \phi(x,0) \\ &\left\{\left(\partial_x^2 - k^2\right)^2 - m_2\omega^2 + 1\right\} \phi_z(x,0) = \omega^2 \phi(x,0). \end{aligned}$$

The mass density terms are now $m_i/\rho l_2$, $i = 1, 2$ and the same notation is uses to avoid the clatter.

Solution using the W-H technique

This section briefly describes the derivation of the solution of the system of equations given in the previous section using the W-H technique shown in (Chung and Fox, 2002). We apply the Fourier transform to the differential equations of ϕ in the respective domains $x < 0$ and $x > 0$, which are defined as

$$(3) \quad \Phi_+(\alpha, z) = \int_0^\infty \phi(x,z)\, e^{\mathrm{i}\,\alpha x} dx, \quad \Phi_-(\alpha, z) = \int_{-\infty}^0 \phi(x,z)\, e^{\mathrm{i}\,\alpha x} dx.$$

We also denote the same transforms of $\phi_z(x,0)$ in $x > 0$ and $x < 0$ by $\Phi'_\pm(x)$, respectively. From the transforms of Laplace's equation and the condition at $z = -H$, we have

$$(4) \quad \Phi_\pm(\alpha, z) = \Phi_\pm(\alpha, 0) \frac{\cosh\gamma(z+H)}{\cosh\gamma H} \pm g(\alpha, z)$$

where $\gamma = \sqrt{\alpha^2 + k^2}$ and $g(\alpha, z)$ is a regular function determined by $\{\phi_x(0,z) - \mathrm{i}\,\alpha\phi(0,z)\}$. Note that $\mathrm{Re}\,\gamma > 0$ when $\mathrm{Re}\,\alpha > 0$ and $\mathrm{Re}\,\gamma < 0$ when $\mathrm{Re}\,\alpha < 0$. Note that the functions $\Phi_\pm(\alpha, z)$ are regular in $\mathrm{Im}\,\alpha > 0$ and $\mathrm{Im}\,\alpha < 0$, respectively. We have, by differentiating the both sides of Eq. 4 with respect to z at $z = 0$,

$$(5) \quad \Phi'_\pm(\alpha) = \Phi_\pm(\alpha, 0)\, \gamma \tanh\gamma H \pm g_z(\alpha, 0)$$

where $\Phi'_{\pm}(\alpha)$ is the z-derivative of $\Phi_{\pm}(\alpha, z)$ at $z = 0$.

Applying the same integral transforms given by Eq. 3 to the plate equations in $x < 0$ and $x > 0$ and from Eq. 5, we obtain

$$
\begin{aligned}
&(6) \quad f_1(\gamma)\,\Phi'_-(\alpha) + C_1(\alpha) = 0 \\
&(7) \quad f_2(\gamma)\,\Phi'_+(\alpha) + C_2(\alpha) = 0
\end{aligned}
$$

where

$$
f_1(\gamma) = l_{\mathrm{r}}^4\gamma^4 - m_1\omega^2 + 1 - \frac{\omega^2}{\gamma\tanh\gamma H}\quad,\qquad f_2(\gamma) = \gamma^4 - m_2\omega^2 + 1 - \frac{\omega^2}{\gamma\tanh\gamma H},
$$

$$
C_1(\alpha) = -\frac{\rho\omega^2 g_z(\alpha,0)}{\gamma\tanh\gamma H} + P_1(\alpha)\quad,\qquad C_2(\alpha) = \frac{\rho\omega^2 g_z(\alpha,0)}{\gamma\tanh\gamma H} - P_2(\alpha).
$$

and P_j, $j = 1,2$, are second order polynomials of α. Functions f_1 and f_2 are called *dispersion* functions and zeros of these functions are the primary tools of our method of deriving the solutions. How to compute the zeros of the dispersion functions are given in (Fox and Chung, 2002).

Functions $\Phi'_-(\alpha,0)$, and $\Phi'_+(\alpha,0)$ are defined in $\operatorname{Im}\alpha < 0$ and $\operatorname{Im}\alpha > 0$ respectively, however they can be extended in the whole plane defined by Eqs. 6 and 7 using the analytic continuation. Eqs. 6 and 7 show that the singularities of Φ'_- and Φ'_+ are determined by the positions of the zeros of f_1 and f_2, since $g_z(\alpha,0)$ is bounded and zeros of $\gamma\tanh\gamma H$ are not the singularities of $\Phi'_{\pm}$. At the moment the two functions $\Phi'_-(\alpha)$ and $\Phi'_+(\alpha)$ do not share the domains of regularity in the upper and the lower half planes separated by the real axis. We are able to manipulate the region of the regularity by adding or subtracting one (or more) singular part of the functions. Here, we subtract a singular part corresponding to $-\lambda$ from $\Phi'_-(\alpha)$ and from $\Phi'_+(\alpha)$, that is, denoting the modified functions $\Psi'_{\pm}(\alpha)$, we have

$$
\begin{aligned}
&(8) \quad f_1(\gamma)\,\Psi'_-(\alpha) + C_1(\alpha) = 0, \\
&(9) \quad f_2(\gamma)\,\Psi'_+(\alpha) - \frac{I f_2(\lambda)}{\alpha+\lambda} + C_2(\alpha) = 0.
\end{aligned}
$$

Then, considering infinitesimal dissipation and the time factor $\exp(+\mathrm{i}\,\omega t)$, the functions $\Psi'_-(\alpha)$ and $\Psi'_+(\alpha)$ are regular on the real axis, or more precisely, the real axis indented over the negative real wave numbers $-\lambda$, $-\mu$ and indented under the positive real wavenumbers λ, μ, respectively. Let $\mathcal{D}_+$ and $\mathcal{D}_-$ denote the upper and lower half of the α-plane that are sharing the indented region on the real axis described above and shown in Fig. 3. Then, we may now add the both sides of Eqs. 8 and 9 to derive a typical W-H equation which is defined on $\mathcal{D}_+ \cap \mathcal{D}_-$,

$$
(10) \quad f_1(\gamma)\,\Psi'_-(\alpha) + f_2(\gamma)\,\Psi'_+(\alpha) - \frac{I f_2(\lambda)}{\alpha+\lambda} + C(\alpha) = 0
$$

where $C = C_1 + C_2$.

We now follow the standard method of solution of the W-H equation, which requires the factorization of the two dispersion functions. We have from Weierstrass' factor theorem and the symmetry of the positions of the roots in S_1 and S_2 (sets of roots of f_1 and f_2 in $\mathcal{D}_+$, respectively),

$$\frac{f_2}{f_1} = K(\alpha) K(-\alpha), \quad K(\alpha) = \left(\prod_{q \in S_1} \frac{q'}{q+\alpha}\right)\left(\prod_{q \in S_2} \frac{q+\alpha}{q'}\right).$$

Note that $K(\alpha)$ is regular in the upper half plane and on the real axis except at $-\lambda$ and $-\mu$ and the infinite products converge in the order of q^{-5} as $|q|$ becomes large. Then Eq. 10 can be rewritten as

$$\begin{aligned} &K(\alpha)\left[f\Psi'_+ + C\right] - \left(K(\alpha) - \tfrac{1}{K(\lambda)}\right)\tfrac{f_2(\lambda')I}{\alpha+\lambda} \\ &= -\tfrac{1}{K(-\alpha)}\left[f\Psi'_- - C\right] - \left(\tfrac{1}{K(-\alpha)} - \tfrac{1}{K(\lambda)}\right)\tfrac{f_2(\lambda')I}{\alpha+\lambda}. \end{aligned} \tag{11}$$

where $f(\gamma) = f_2(\gamma) - f_1(\gamma)$. Note that the splitting is actually performed on $f_1\gamma\sinh\gamma H$ and $f_2\gamma\sinh\gamma H$ which are none zero at $\gamma = 0$.

The left hand side of Eq. 11 is regular in $\mathcal{D}_+$ and the right hand side is regular in $\mathcal{D}_-$. Notice that a function is added to the both sides of the equation to make the right hand side of the equation regular in $\mathcal{D}_-$. The left hand side of Eq. 11 is $o(\alpha^4)$ as $|\alpha| \to \infty$ in $\mathcal{D}_+$, since $\Psi'_+ \to 0$ and $K(\alpha) = O(1)$ as $|\alpha| \to \infty$ in $\mathcal{D}_+$. And the right hand side of Eq. 11 has the same analytic properties in $\mathcal{D}_-$. The Liouville's theorem in section 2.4 of (Carrier et al, 1966) tells us that there exists a function, denoted by $J(\alpha)$, uniquely defined by Eq. 11, and function $J(\alpha)$ is a polynomial of degree three in the whole plane

$$J(\alpha) = d_0 + d_1\alpha + d_2\alpha^2 + d_3\alpha^3.$$

Equating Eq. 11 for $\Psi' = \Psi'_- + \Psi'_+$ gives

$$\Psi'(\alpha) = \frac{-F(\alpha)}{K(\alpha) f_1(\gamma)} \quad \text{or} \quad -\frac{K(-\alpha) F(\alpha)}{f_2(\gamma)} \tag{12}$$

where

$$F(\alpha) = J(\alpha) - \frac{I f_2(\gamma)}{(\alpha+\lambda) K(\lambda)}.$$

Notice that procedure in Eq. 11 eliminates the need of calculating C. We are now able to calculate using the following inverse Fourier transform

$$\phi_z(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi'(\alpha) e^{-\mathrm{i}\alpha x}\, d\alpha.$$

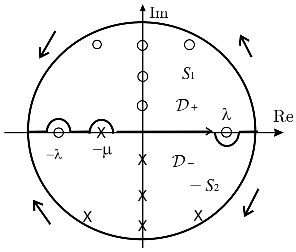

Figure 3: Schematics of the contour integration paths of the inverse Fourier transform and the positions (not to scale) of the corresponding roots in the α-plane. μ is not shown since it is not a singularity of the integrand.

For $x<0$ we close the integral contour in $\mathcal{D}_+$ as depicted in Fig. 3, and put the incident wave back, then we have

$$(13)\quad \phi_z(x,0) = \mathrm{i}\, I e^{\mathrm{i}\,\lambda x} - \sum_{q\in S_1} \frac{\mathrm{i}\, F(q)\, q' R_1(q')}{qK(q)} e^{-\mathrm{i}\, qx},$$

$$(14)\quad R_1(q') = \left(\left. \frac{df_1(\gamma)}{d\gamma} \right|_{\gamma=q'} \right)^{-1} = \frac{\omega^2 q'}{(5l_\mathrm{r}^4 q'^4 + b_1)\,\omega^2 + H\left((l_\mathrm{r}^4 q'^5 + b_1 q)^2 - \omega^4 \right)},$$

where $R_1(q')$ is a residue of $[f_1(\gamma)]^{-1}$ at $\gamma = q'$. We used $b_1 = -m_1\omega^2 + 1$ and $f_1(q') = 0$ to simplify the formula. The velocity potential $\phi(x,z)$ is, from Eqs. 4 and 5,

$$\phi(x,z) = \frac{\mathrm{i}\, I \cosh \lambda'(z+H)}{\lambda' \sinh \lambda' H} e^{\mathrm{i}\,\lambda x} - \sum_{q\in S_1} \frac{\mathrm{i}\, F(q)\, R_1(q') \cosh q'(z+H)}{qK(q) \sinh q' H} e^{-\mathrm{i}\, qx}$$

where $\lambda' = \sqrt{\lambda^2 + k^2}$. And for $x > 0$, we derive $\phi_z(x,0)$ then $\phi(x,z)$ by closing the integral contour in $\mathcal{D}_-$ as depicted in Fig. 3,

$$(15)\quad \begin{aligned} \phi_z(x,0) &= -\sum_{q\in S_2} \frac{\mathrm{i}\, K(q)\, F(-q)\, q' R_2(q')}{q} e^{\mathrm{i}\, qx}, \\ \phi(x,z) &= -\sum_{q\in S_2} \frac{\mathrm{i}\, K(q)\, F(-q)\, R_2(q') \cosh q'(z+H)}{q \sinh q' H} e^{\mathrm{i}\, qx}, \end{aligned}$$

where R_2 is a residue of $[f_2(\gamma)]^{-1}$ and its formula can be obtained by replacing the subscript 1 with 2 and l_r with 1 in Eq. 14. Notice that from $R_j \sim O(q^{-9})$ coefficients of ϕ_z decay as $O(q^{-6})$ as $|q|$ becomes large, thus the displacement is bounded up to the fourth x-derivatives. The coefficients of ϕ, have extra $1/q' \tanh q' H$ term which is $O(q^4)$, hence the coefficients decay as $O(q^{-2})$ as $|q|$ becomes large. Therefore, ϕ is bounded everywhere including at $x = 0$.

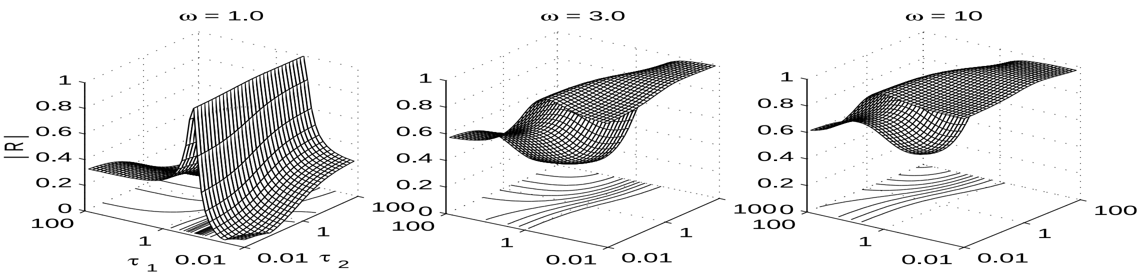

Figure 4: The reflection coefficients and the spring constants. The axes τ_1 and τ_2 are in log-scale.

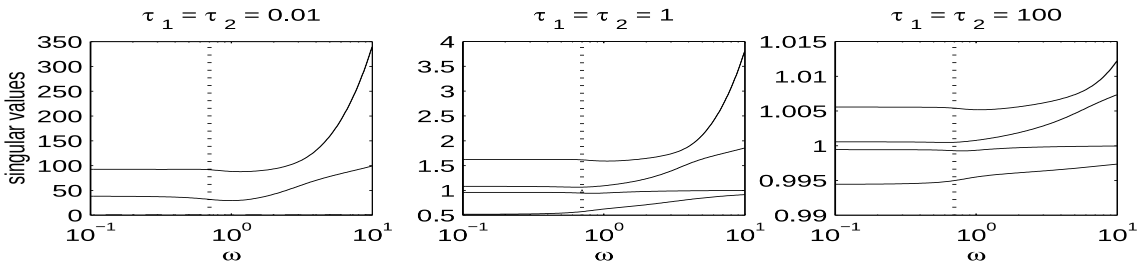

Figure 5: The singular values of the transition matrix for various spring constants. The dotted line is at $\omega = 0.7$.

TRANSITION CONDITIONS

The four coefficients of $J(\alpha)$ are determined by physical conditions at $x = 0$, which we call transition conditions. We consider an elastic joining material at the transition. By changing the elasticity and the thickness of the joining material, the transmission of waves can be studied as function of the transition conditions.

The transition conditions are given by the scaled effective shear force,

$$(16)\quad B_1 w|_{x=0+} = w_{xxx} - k^2(2-\nu)\, w_x$$

$$(17)\quad B_1 w|_{x=0-} = l_{\mathrm{r}}^4 \left\{ w_{xxx} - k^2(2-\nu)\, w_x \right\}$$

and the scaled bending moment

$$(18)\quad B_2 w|_{x=0+} = w_{xx} - k^2 \nu w$$

$$(19)\quad B_2 w|_{x=0-} = l_{\mathrm{r}}^4 \left(w_{xx} - k^2 \nu w \right)$$

Since all possible transitions conditions are linear equations with respect to w, the transition conditions can be expressed by algebraic equations of column vector $\mathbf{d} = (d_0, d_1, d_2, d_3)$ made of the coefficients of polynomial J.

$$\begin{aligned} w|_{x=0\pm} &= \mathcal{A}^{\pm} \cdot \mathbf{d} + \mathcal{B}^{\pm},\ \ w_x|_{x=0\pm} = \mathcal{C}^{\pm} \cdot \mathbf{d} + \mathcal{D}^{\pm}, \\ B_1 w_x|_{x=0\pm} &= \mathcal{E}^{\pm} \cdot \mathbf{d} + \mathcal{F}^{\pm},\ \ B_2 w_x|_{x=0\pm} = \mathcal{G}^{\pm} \cdot \mathbf{d} + \mathcal{H}^{\pm}, \end{aligned}$$

where $\mathcal{A}^{\pm}$, $\mathcal{C}^{\pm}$, $\mathcal{E}^{\pm}$ and $\mathcal{G}^{\pm}$ are row vectors, and $\mathcal{B}^{\pm}$, $\mathcal{D}^{\pm}$, $\mathcal{F}^{\pm}$ and $\mathcal{H}^{\pm}$ are scalar values that are calculated from Eqs. 13, 15, and $\cdot$ is the vector inner product.

When the two plates have the same thickness, i.e., $l_{\mathrm{r}} = 1$ and $k = 0$. Note that there is no need for factorization since, $K \equiv 1$. The incident wave does not automatically appear from the inverse Fourier transform in this case since the wave number for incident wave and traveling wave are the same. Hence, Eq. 13 must be changed to

$$\phi_z(x, 0) = \mathrm{i}\, I e^{\mathrm{i}\,\lambda x} - \sum_{q \in S_1} \frac{\mathrm{i}\, J(-q)\, q' R_2(q')}{q} e^{\mathrm{i}\,qx}.$$

We consider the transition linked with two springs in the vertical and the rotational directions as shown in Fig. 2. Thus, the bending moment at the transition is determined by the rotational spring constant and the difference of the gradient of the two ice sheets. The shear force at the transition is determined by the vertical spring constant and the difference of the vertical displacement of the ice sheets. From the equilibrium of the force, the bending moment and the shear force are continuous at the transition. We thus have the following equations,

$$B_1 w_1 = B_1 w_2, \quad B_2 w_1 = B_2 w_2. \tag{20}$$

From the argument above, the displacement and the gradient are coupled with the above equations by the following equations

$$\begin{aligned} \tau_1 \left(w|_{x=0-} - w|_{x=0+} \right) &= \pm\, B_1 w|_{x=0\pm}\,, \\ \tau_2 \left(w_x|_{x=0-} - w_x|_{x=0+} \right) &= \pm\, B_2 w|_{x=0\pm}\,, \end{aligned} \tag{21}$$

where τ_1 and τ_2 are the rotational and the vertical spring constants, respectively. Equations 20 and 21 can then be expressed by the following matrix equation,

$$\mathcal{T}\, \mathbf{d} = \mathbf{v}$$

where $\mathcal{T}$ and $\mathbf{v}$ are the transition matrix and the corresponding vector, respectively.

At a low frequency range the variation of the transitions conditions makes little difference to the wave propagation. Fig. 5 shows 4 singular values of $\mathcal{T}$. The figures show that the transition matrix stays virtually unchanged for $\omega < 0.7$ regardless of varying spring constants. Fig. 4 shows that the vertical and the rotational springs are equally influential to the reflection of the waves. However, the vertical spring induces more rapid variation to the reflection.

CONCLUSIONS

This paper gives an initial attempt at modelling the transition conditions in an ice sheet. The conditions are reduced to the vertical and the rotational springs. Further studies and field measurements must be conducted to find the correlation between the theoretical model here and the real transition in an ice sheet. The fact that the solution by the W-H technique has explicit formulae (invariant to the transition conditions) for the coefficients of d_0, d_1, d_2, d_3 is exploited here to study exclusively the transition conditions. The wave propagation is highly dependent on the transition conditions. It is shown that a small change in the spring constants results in a large fluctuation in the reflection of the waves.

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