# Vector Green's Functions for Electrodynamics Applications

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***Abstract*—The use of scalar Green's functions is commonplace in electrodynamics, but many useful systems require computation of one or more vector quantities. Vector Green's functions have been used in electrodynamics for many decades, but are inconsistent and poorly understood. We recast vector Green's functions in the language of distribution theory to match their scalar counterparts. We use vector Green's functions to calculate the electrodynamic vector potential under physical boundary conditions.**

## I. Introduction

Green's functions have simplified the solving of inhomogeneous, linear, scalar boundary value problems (BVPs) which are common in many fields of study, e.g. quantum physics [1], many body simulations [2], and electrodynamics [3]. If one can find the Green's function for a BVP, then one can construct the solution to an arbitrary region through use of the boundary element method [4].

While Green's functions are very old, it was only the development of distribution theory and the notion of a generalised function that allowed a full understanding of the uses and behaviour of Green's functions [5]. Modern distribution theory deals with scalar functions and functionals. Despite the development of the theory of vector distributions shortly thereafter [6], it passed largely unnoticed.

Vector Green's functions have been used for the better part of a century, but their use has not generally been rigorously justified. However the selections of the functions are inconsistent [7] [8]. This can be put on a secure footing by a connection to distribution theory and generalised functions such as that which allowed scalar Green's functions to become so ubiquitous. Moreover, vector Green's functions are often constructed such that they satisfy radiation conditions in order to guarantee uniqueness of the solutions to the BVPs. While this does yield useful solutions, radiation conditions are just a small fraction of possible physically relevant boundary conditions. Thus, only developing theory for radiation conditions at the boundary limits possible applications.

In this paper we connect the existing vector distribution theory to create a general theory for vector Green's functions. We then develop vector Green's functions for specific cases for the electrodynamic vector potential with simple examples, finding analogies to common scalar boundary conditions in order to guarantee the uniqueness of the electric and magnetic fields.

## II. Vector Green's Functions

### A. Background

The following are some useful definitions from Ref [6].

**Definition II.1.** A *test function* is an infinitely differentiable function with compact support. This is also often referred to as a *bump function*.

**Definition II.2.** A *functional*, $< f, \cdot >$, is defined such that for each function $\phi(x)$, it assigns a real number. We will use functionals of the form:

$$< f, \phi \underset{\Omega}{>} \stackrel{def}{=} \int_{\Omega} f(\mathbf{x})\phi^*(\mathbf{x})dV, \tag{1}$$

for a given function $f(\mathbf{x})$ in the region $\Omega$ with the star denoting the complex conjugate.

**Definition II.3.** A *distribution* is a continuous linear functional on the space of test functions. The distribution generated by the function $f(\mathbf{x})$ is denoted $f$.

Now let $f(\mathbf{x})$ be differentiable and $f^{(i)}(\mathbf{x})$ locally integrable, using the superscript $(i)$ to denote a single differentiation in the $x_i$ variable. The distribution generated by $f^{(i)}(\mathbf{x})$, found through integration by parts is:

$$< f^{(i)}, \phi \underset{\Omega}{>} = - < f, \phi^{(i)} \underset{\Omega}{>} \tag{2}$$

**Definition II.4.** A *test vector* is defined with test functions $t_1(\mathbf{x})$, ..., $t_n(\mathbf{x})$ as

$$\mathbf{T}(\mathbf{x}) = \hat{e}_1 t_1(\mathbf{x}) + ... + \hat{e}_n t_n(\mathbf{x}), \tag{3}$$

where $\hat{e}_i$ is the $i$th basis unit vector.

**Definition II.5.** A vector $\mathbf{S}$ whose components are distributions $s_1$, ..., $s_n$ is called a *vector distribution*, i.e.,

$$\mathbf{S} = \hat{e}_1 s_1 + ... + \hat{e}_n s_n. \tag{4}$$

We can now extend our functionals to deal with vectors by:

$$\begin{aligned} < \mathbf{S}, \mathbf{T} > &= < s_1, t_1 > + < s_2, t_2 > + < s_3, t_3 > \\ &= \int \mathbf{S} \cdot \mathbf{T}^* dV \end{aligned} \tag{5}$$

The identities for the gradient, divergence, and curl follow from equation (2):

$$< \nabla s, \mathbf{T} > = - < s, \nabla \cdot \mathbf{T} >, \tag{6}$$

$$< \nabla \cdot \mathbf{S}, t > = - < \mathbf{S}, \nabla t >, \tag{7}$$

$$< \nabla \times \mathbf{S}, \mathbf{T} > = - < \mathbf{S}, \nabla \times \mathbf{T} > . \tag{8}$$

*B. Scalar Green's functions*

The following is a brief summary of scalar Green's functions [5].

Any linear BVP in a region $\Omega$ can be expressed as

$$Lu(\mathbf{x}) = f(\mathbf{x}), \quad B_{(i)}(u) = c_i \tag{9}$$

where $L$ is a linear partial differential operator, $f(\mathbf{x})$ a function describing the inhomogeneity, and $B_{(i)} = c_j$ is the $j$'th boundary condition (BC) over the boundary $\partial\Omega$.

**Definition II.6.** The *Green's function*, $g(\mathbf{x}|\boldsymbol{\xi})$, is the solution with compact support to the related homogeneous partial differential equation (PDE),

$$Lg(\mathbf{x}|\boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}), \quad B_{(i)}(u) = 0, \tag{10}$$

where $\boldsymbol{\xi} = (x'_1, ..., x'_n)$.

If the boundary is at infinity, then the solution to the BVP is simply

$$u(\mathbf{x}) = \int\limits_{\Omega} g^*(\mathbf{x}|\boldsymbol{\xi}) f(\boldsymbol{\xi}) d^n\boldsymbol{\xi} \tag{11}$$

due to the compact support condition, where $n$ is the number of dimensions. Otherwise we must look at the adjoint problem:

$$L^* w(\mathbf{x}|\boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}), \quad B^*_{(i)}(w) = 0, \tag{12}$$

where $w(\mathbf{x}|\boldsymbol{\xi})$ is the adjoint Green's function and $B^*_{(i)}$ are the adjoint BCs.

Then, substituting (12) and (9) into (1) we integrate by parts to find

$$< Lu, w \underset{\Omega}{>} = < u, L^* w \underset{\Omega}{>} + \Big[ J(u, w) \Big]_{\Omega}, \tag{13}$$

where J is referred to as the bilinear concommitant. Since $L^* w(\mathbf{x}|\boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi})$ and $Lu(\mathbf{x}) = f(\mathbf{x})$, we find that

$$u(\mathbf{x}) = \int\limits_{\Omega} f(\boldsymbol{\xi}) w^*(\mathbf{x}|\boldsymbol{\xi}) d^n\boldsymbol{\xi} + \Big[ J(u, w) \Big]_{\Omega} \tag{14}$$

We can choose the adjoint boundary conditions however we like, so it makes sense to choose them such that $\Big[ J(u, w) \Big]_{\Omega}$ is as simple as possible. Ideally this can be made to vanish.

*1) Electrodynamics Application:* These techniques have been very useful in solving Maxwell's equations for the scalar potential, $\Phi$. Maxwell's equations for a static charge density, $\rho(\mathbf{x})$, in free space with no magnetic fields can be rewritten as Poisson's equation

$$\nabla^2 \Phi(\mathbf{x}) = -\epsilon_0^{-1} \rho(\mathbf{x}). \tag{15}$$

In free space the boundary conditions vanish and we can find the solution to this through equation (11) as the system is self adjoint. The Green's function is defined by

$$\nabla^2 g(\mathbf{x}|\boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}), \tag{16}$$

which can be solved in 3-D to yield:

$$g(\mathbf{x}|\boldsymbol{\xi}) = -\frac{1}{4\pi|\mathbf{x} - \boldsymbol{\xi}|}, \tag{17}$$

where $|\mathbf{x} - \boldsymbol{\xi}| = ((x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2)^{\frac{1}{2}}$. This yields the well known electrostatic potential for a point charge

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|} d^3\boldsymbol{\xi}. \tag{18}$$

The free space Green's function can be used for problems where the boundaries do not extend to infinity by finding uniqueness conditions based on the boundary of a region. This allows some complex boundaries and charge distributions to be solved for through the method of images and boundary element calculations [4].

*2) Vector Green's function:* A vector BVP over the region $\Omega$ can be described in a similar way to the scalar problem. In general they will take the form:

$$\mathbf{Lu}(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \quad \mathbf{b}_{(i)}(\mathbf{u}) = \mathbf{c}_i \tag{19}$$

for some vector linear differential operator $\mathbf{L}$ and boundary conditions $\mathbf{b}_{(i)}$ on the boundary $\partial\Omega$. We start our development of vector Green's functions by defining its adjoint, $\mathbf{W}_1(\mathbf{x}|\boldsymbol{\xi})$, as the solution to the homogeneous adjoint BVP:

$$\mathbf{L}^* \mathbf{W}_1(\mathbf{x}|\boldsymbol{\xi})(\mathbf{x}) = \delta(\mathbf{x} - \boldsymbol{\xi})\hat{e}_1, \quad \mathbf{b}^*_{(i)}(\mathbf{W}_1) = 0. \tag{20}$$

Using our vector distribution theory it can be shown that

$$< \mathbf{Lu}, \mathbf{W}_1 \underset{\Omega}{>} = < \mathbf{u}, \mathbf{L}^* \mathbf{W}_1 \underset{\Omega}{>} + \oint\limits_{\partial\Omega} \mathbf{C}_1 \cdot \hat{\mathbf{n}}\, ds \tag{21}$$

Where $\mathbf{L}^*$ is the adjoint linear operator and $\mathbf{C}_1$ is the vector bilinear concommitant. By using (19) and (20), this can be rewritten as

$$\mathbf{u} \cdot \hat{e}_1 = -\int\limits_{\Omega} \mathbf{W}^*_1 \cdot \mathbf{f}\, d^n\boldsymbol{\xi} - \oint\limits_{\partial\Omega} \mathbf{C}_1 \cdot \hat{\mathbf{n}}\, ds. \tag{22}$$

Evidently, this Green's function yields the $\hat{e}_1$ component of the solution. The same process can be done for all basis vectors, and hence the full solution can be expressed as:

$$\begin{aligned} \mathbf{u} &= \sum_{j=1}^{n} \big( \mathbf{u} \cdot \hat{e}_1 \big) \\ &= \sum_{j=1}^{n} \Bigg( -\int\limits_{\Omega} \mathbf{W}^*_j \cdot \mathbf{f}\, d^n\boldsymbol{\xi} - \oint\limits_{\partial\Omega} \mathbf{C}_j \cdot \hat{\mathbf{n}}\, ds \Bigg), \end{aligned} \tag{23}$$

where the subscript $j$ refers to the fact that the Green's function and bilinear concommitant are generated by the directional delta function $\delta(\mathbf{x} - \boldsymbol{\xi})\hat{e}_j$. This can be rewritten by collecting the $\mathbf{W}_j$ and $\mathbf{C}_j$'s into matrices, where each $j$ denotes a row. Thus equation (23) can be written simply as:

$$\mathbf{u} = -\int_{\Omega} \mathbb{W}^* \mathbf{J}\, d^n\boldsymbol{\xi} - \oint_{\partial\Omega} \mathbb{C}\hat{\mathbf{n}}\, ds, \tag{24}$$

with the block letters denoting rank 2 tensors (a.k.a. matrices/dyadics).

The BVPs that shall be covered in this paper are self adjoint. This means that the BVP is the same as its adjoint and hence the Green's function is also self adjoint so we can write $\mathbb{W} = \mathbb{G}$, where $\mathbb{G}$ is the vector Green's function, constructed from $\mathbf{G}_i$'s and the adjoint of $\mathbb{W}$.

## III. Specific Solutions for Electrodynamics

### A. Free Space Magnetostatic Vector Potential

The 3-D static vector potential generated by a current distribution $\mathbf{J}(\mathbf{x})$ is governed by the Vector Poisson's equation in the Lorentz gauge [3]:

$$\nabla^2 \mathbf{A}(\mathbf{x}) = -\mu_0\, \mathbf{J}(\mathbf{x}), \tag{25}$$

where the region is unbounded. We choose the solutions such that $\mathbf{A}$, and it's derivatives go to zero as $|\mathbf{x}| \to \infty$, hence the bilinear concomitant vanishes. Substituting this into equation (22), we obtain:

$$\mathbf{A} \cdot \hat{x} = -\int \mu_o\, \mathbf{G}_x^*(\mathbf{x}|\boldsymbol{\xi}) \cdot \mathbf{J}(\boldsymbol{\xi})\, d^3\boldsymbol{\xi}. \tag{26}$$

We find $\mathbf{G}_x$ by solving the PDE $\nabla^2 \mathbf{G}_x(\mathbf{x}|\boldsymbol{\xi}) = \delta(\mathbf{x}-\boldsymbol{\xi})\hat{x}$, which can be done through separation of variables. However, because the function must go to zero at infinity, the components orthogonal to the current source have only the trivial solution. Thus,

$$\mathbf{G}_x(\mathbf{x}|\boldsymbol{\xi}) = -\frac{1}{4\pi|\mathbf{x}-\boldsymbol{\xi}|}\hat{x}. \tag{27}$$

By repeating this for $\hat{y}$ and $\hat{z}$,

$$\mathbb{G}(\mathbf{x}|\boldsymbol{\xi}) = g(\mathbf{x}|\boldsymbol{\xi})\, \mathbb{I}, \tag{28}$$

where $\mathbb{I}$ is the identity matrix and $g(\mathbf{x}|\boldsymbol{\xi})$ is the free space scalar potential Green's function from (17).

The vector potential for free space is then given by:

$$\mathbf{A}(\mathbf{x}) = -\mu_o \int \mathbb{G}^*(\mathbf{x}|\boldsymbol{\xi})\, \mathbf{J}(\boldsymbol{\xi})\, d^3\boldsymbol{\xi}. \tag{29}$$

This equation can be found in most sources on dyadic Greens functions [9], but is often assumed with, at best, only a loose justification. We have showed that the approach is justified in this case.

Recalling that $\mathbb{G}^* = \mathbb{G}$ and that it is diagonal, we obtain the well known equation for vector potential:

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_o}{4\pi} \int \frac{\mathbf{J}(\boldsymbol{\xi})}{|\mathbf{x}-\boldsymbol{\xi}|}\, d^3\boldsymbol{\xi}. \tag{30}$$

### B. Half-Space Magnetostatic Vector Potential

A current source sits somewhere in region $\Omega_1$ above a boundary $\partial\Omega$ defined by the plane $z = 0$. The regions $\Omega_1 = \{z > 0\}$ is vacuum and $\Omega_2 = \{z < 0\}$ is linear with respective permativities and permiabilities $\epsilon_2$ and $\mu_2$. There are no charge distributions.

First we need find conditions for the uniqueness of the magnetic field. Let us define two solutions to an equivalent version of equation (25),

$$\nabla \times \nabla \times \mathbf{A}(\mathbf{x}) = -\mu_0\, \mathbf{J}(\mathbf{x}), \tag{31}$$

as $\mathbf{A}_1$ and $\mathbf{A}_2$, for some $\mathbf{J}(\mathbf{x})$. The difference $\mathcal{A} = \mathbf{A}_1 - \mathbf{A}_2$ then satisfies the equation:

$$\nabla \times \nabla \times \mathcal{A} = 0. \tag{32}$$

Taking the dot product between this with $\mathcal{A}$ then integrating over a region $\Omega$ gives:

$$\int_{\Omega} (\mathcal{A} \cdot \nabla \times \nabla \times \mathcal{A})\, dV = \tag{33}$$

$$\int_{\Omega} \Big( (\nabla \times \mathcal{A}) \cdot (\nabla \times \mathcal{A}) - (\nabla \cdot \mathcal{A} \times \nabla \times \mathcal{A}) \Big)\, dV.$$

The LHS vanishes and we apply the divergence theorem to obtain:

$$\int_{\Omega} (\nabla \times \mathcal{A})^2\, dV = \oint_{\partial\Omega} (\mathcal{A} \times \nabla \times \mathcal{A}) \cdot \hat{\mathbf{n}}\, ds. \tag{34}$$

Since $(\nabla \times \mathcal{A}(\mathbf{x}))^2 > 0\ \forall \mathbf{x}$ then $\nabla \times \mathcal{A}(\mathbf{x})$, and hence the difference in magnetic fields between the two solutions will be zero only if the surface integral is zero. Therefore any magnetic field produced by an $\mathbf{A}$ that satisfies our BVP will be unique. There are several obvious cases where this will be true, which can be connected to scalar analogies as presented in [4]:

1) Dirichlet-esque boundary condition: Where $\mathbf{A}$, or $\hat{\mathbf{n}} \times \mathbf{A}$ is well defined along on $\partial\Omega$.
2) Neumann-esque boundary condition: Where $\nabla \times \mathbf{A}$, or $\hat{\mathbf{n}} \times \nabla \times \mathbf{A}$ is well defined on $\partial\Omega$.
3) Mixed or Robin-esque boundary condition: Where a linear combination of the above four terms is well defined on $\partial\Omega$.

Our boundary condition is informed by the fact that no current can cross $\partial\Omega$. Using Lorentz's force law [3] for some arbitrary vector $\mathbf{v}$,

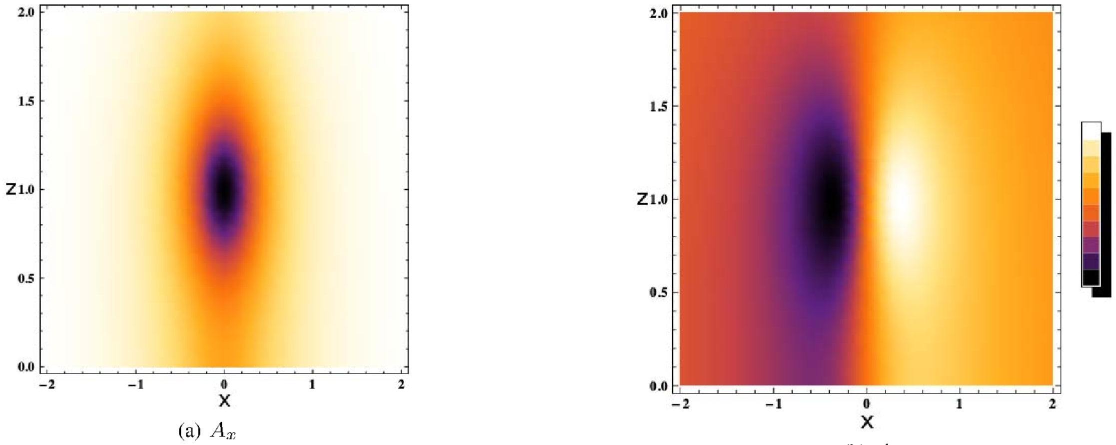
$$\hat{e}_z \cdot \big[\mathbf{E} + \mathbf{v} \times \mathbf{B}\big]_{z=0} = 0. \tag{35}$$

But since $\mathbf{v}$ is arbitrary this only holds when $[\hat{e}_z \times \mathbf{B}]_{z=0} = 0$ and $[E_z]_{z=0} = 0$, with the latter trivially satisfied. Rewriting in terms of the vector potential, the BC can be expressed

$$\hat{\mathbf{n}} \times \nabla \times \mathbf{A} = 0, \tag{36}$$

a BC of the second type listed above.

In addition to this we require that our fields and their derivatives tend to zero as $|\mathbf{x}| \to \infty$.




(a) $A_x$

(b) $A_y$

Fig. 1: Vector potential at $y = 0.5$ generated by a small loop of current with $\rho_0 = 0.1$, centered at $z = 1$ in a half space $z > 0$. The scale and constants are chosen arbitrarily.

No looking at the vector Green's functions, $\mathbf{G}_i$, we firstly note that they are vector potentials themselves and therefore also restricted by condition (36). We assume that the tensor $\mathbb{G}$ is diagonal and substitute its components into (36) to obtain simplified BC's

$$\frac{\partial}{\partial z}\mathbf{G}_x\Big|_{z=0} = 0, \quad \frac{\partial}{\partial z}\mathbf{G}_y\Big|_{z=0} = 0, \tag{37}$$

$$\frac{\partial}{\partial x}\mathbf{G}_z\Big|_{z=0} = 0, \quad \frac{\partial}{\partial y}\mathbf{G}_z\Big|_{z=0} = 0. \tag{38}$$

The Green's functions here are written as vectors, but since two of their components are zero, we can treat them as scalar functions with well known solutions.

We shall find the vector Green's functions using the method of images, whereby the boundary conditions can be met by superimposing the free space solution of a delta source with free space solutions of delta sources located outside the region of interest [4]. Condition (38) is a homogeneous Dirchelet condition, which for an infinite plane the solution is given by superimposing an image delta source of opposite sign the same distance from the boundary, but on the other side as the main source. The conditions in (37) are homogeneous Neumann conditions, satisfied in the same way as above, only with the same sign as the original source [4]. Thus the Green's tensor takes the form:

$$\mathbb{G} = -\frac{1}{4\pi|\mathbf{x}-\boldsymbol{\xi}|}\mathbb{I} - \frac{1}{4\pi|\mathbf{x}-\boldsymbol{\xi}_r|}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{39}$$

where $\boldsymbol{\xi} = (x', y', z')$ and $\boldsymbol{\xi}_r = (x', y', -z')$ with the $r$ denoting reflection. The solution for $\mathbf{A}$ is provided by equation (24) with the surface integral vanishing under our conditions, assuming there is no time varying charge distribution.

An example use of this is an imperfect magnetic dipole, defined by the current distribution

$$\mathbf{J}(\boldsymbol{\xi}) = I\delta(\rho' - \rho_0)\delta(z' - 1)\hat{\boldsymbol{\phi}}', \tag{40}$$

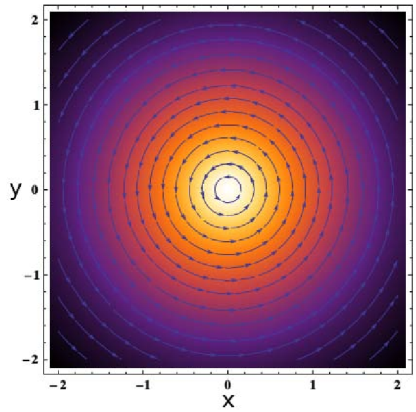



Fig. 2: A cross section of $\mathbf{A}$ at $z = 0.5$.

using cylindrical coordinates. Figures 1 and 2 were found using Mathematica's numerical integration package and plotting with arbitrary values.

### C. Half-Space Harmonic Vector Potential

Now consider the Helmholtz equation [3],

$$(\nabla\times\nabla\times + k^2)\mathbf{A}(\mathbf{x}) = -\mu_0\,\mathbf{J}(\mathbf{x}), \tag{41}$$

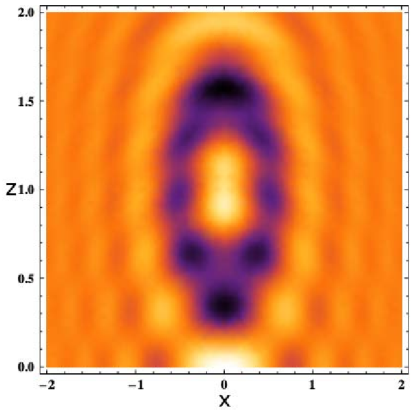
with the same BC's as in the previous section.

Once again we must find uniqueness conditions, which we do by mirroring [10]. We start by defining $\mathcal{A} = \mathbf{A}_1 - \mathbf{A}_2$, where the $\mathbf{A}_i$ are solutions to (41) generated by the Green's function $\mathbf{G}^i_a$, defined by

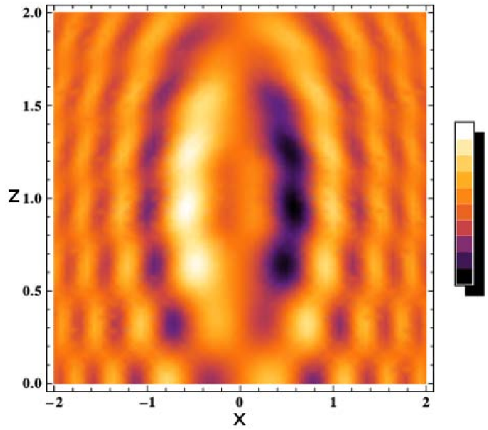
$$(\nabla\times\nabla\times + k^2)\mathbf{G}^i_a = \delta(\mathbf{x}-\boldsymbol{\xi})\hat{\mathbf{a}}, \tag{42}$$

for an arbitrary unit vector $\hat{\mathbf{a}}$. The difference $\mathcal{A}$ satisfies the equation

$$(\nabla\times\nabla\times + k^2)\mathcal{A} = 0. \tag{43}$$




(a) $A_x$




(b) $A_y$

Fig. 3: Vector potential at $y = 0.5$ generated by a small loop of harmonically oscillating current with $\rho_0 = 0.1$, centered at $z = 1$ in a half space $z > 0$. The scale and constants are chosen arbitrarily.

Green's vector identity gives us

$$\int_{\Omega} \left( \mathbf{G}_a^i \cdot (\nabla \times \nabla \times +k^2)\mathcal{A} - \mathcal{A} \cdot (\nabla \times \nabla \times +k^2)\mathbf{G}_a^i \right) dV'$$
$$= \oint_{\partial\Omega} (\mathcal{A} \times \nabla \times \mathbf{G}_a^i - \mathbf{G}_a^i \times \nabla \times \mathcal{A}) \cdot \hat{\mathbf{n}} \, ds'. \qquad (44)$$

We can straight away see that a vector Dirchlet condition of $\hat{\mathbf{n}} \times \mathbf{A}(\mathbf{x}) = -\mathbf{f}(\mathbf{x})$, yields

$$\mathcal{A} \cdot \hat{\mathbf{a}} = \oint_{\partial\Omega} \mathbf{f} \cdot (\nabla \times \mathcal{A}) \, ds', \qquad (45)$$

which is a constant, and therefore creates unique $\mathbf{B}$ and $\mathbf{E}$ fields. Likewise, a vector Neumann condition of $\hat{\mathbf{n}} \times \nabla \times \mathbf{A}(\mathbf{x}) = -\mathbf{f}(\mathbf{x})$, yeilds a constant value of $\mathcal{A} \cdot \hat{\mathbf{a}}$ and creates unique $\mathbf{B}$ and $\mathbf{E}$ fields.

The Robin condition is not as clear. One might try a boundary condition of the form

$$w(\mathbf{x})\mathbf{A}(\mathbf{x}) + \nabla \times \mathbf{A}(\mathbf{x}) = -\mathbf{f}(\mathbf{x}), \text{ or} \qquad (46)$$

$$w(\mathbf{x})\hat{\mathbf{n}} \times \mathbf{A}(\mathbf{x}) + \hat{\mathbf{n}} \times \nabla \times \mathbf{A}(\mathbf{x}) = -\hat{\mathbf{n}} \times \mathbf{f}(\mathbf{x}). \qquad (47)$$

Following the above approach we obtain by using triple cross product identities that

$$\mathcal{A} \cdot \hat{\mathbf{a}} = \oint_{\partial\Omega} \left( (2w\mathbf{G}_a^i + \mathbf{f}) \times \mathcal{A} \right) \cdot \hat{\mathbf{n}} \, ds', \qquad (48)$$

which is not a constant vector as $\mathbf{G}_a^i$ depends on both $\boldsymbol{\xi}$ and $\mathbf{x}$.

On deeper examination we find that there are in fact two Robin-esque conditions. They are:

$$w(\mathbf{x})\mathbf{A}(\mathbf{x}) + \hat{\mathbf{n}} \times \nabla \times \mathbf{A}(\mathbf{x}) = -\mathbf{f}(\mathbf{x}) \text{ and} \qquad (49)$$
$$w(\mathbf{x})\hat{\mathbf{n}} \times \mathbf{A}(\mathbf{x}) + \nabla \times \mathbf{A}(\mathbf{x}) = -\mathbf{f}(\mathbf{x}). \qquad (50)$$

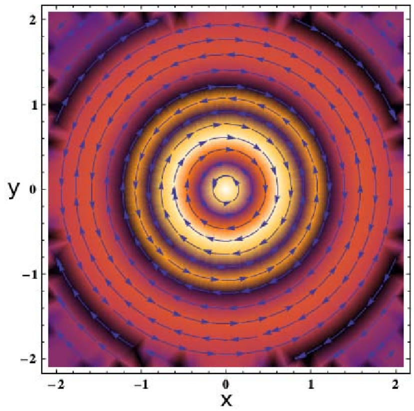



Fig. 4: A cross section of $\mathbf{A}$ at $z = 0.5$. Scale and constants are chosen arbitrarily.

Substituting equation (50) into (44) yields

$$\mathcal{A} \cdot \hat{\mathbf{a}} = \oint_{\partial\Omega} \hat{\mathbf{n}} \cdot \mathbf{f} \times \mathcal{A} \, ds', \qquad (51)$$

which is a constant and creates unique $\mathbf{B}$ and $\mathbf{E}$ fields. The proof for boundary condition (49) is similar.

We have the BC's

$$\hat{\mathbf{n}} \times \nabla \times \mathbf{A} = 0, \text{ and} \qquad (52)$$
$$\left[ \frac{\partial}{\partial t} A_z \right]_{z=0} = 0. \qquad (53)$$

We now know (52) guarantees the uniqueness of the electric and magnetic fields. Condition (53) is a further restriction for when we add the time dependence of $\mathbf{A}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}) \exp(i\omega t)$. This can only happen when $A_z(\mathbf{x}, t)\big|_{z=0} = 0$.

Because of our uniqueness theorems, we can once again use the method of images. The Green's functions are generated in the same way as in the previous section, only with the free space scalar Green's function of Helmholtz equation. That is, our Green's tensor is given by:

$$\mathbb{G} = -\frac{e^{-ik|\mathbf{x}-\boldsymbol{\xi}|}}{4\pi|\mathbf{x}-\boldsymbol{\xi}|}\mathbb{I} - \frac{e^{-ik|\mathbf{x}-\boldsymbol{\xi}_r|}}{4\pi|\mathbf{x}-\boldsymbol{\xi}_r|}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (54)$$

This already satisfies BC (53) as any $A_z$ generated by this will be zero at the boundary. Once again, the $\mathbf{A}$ field is generated by equation (24) and the surface integral vanishes under our conditions, assuming there is no time varying charge distribution.

Now we compute the field of an imperfect oscillating magnetic dipole, defined by the current distribution

$$\mathbf{J}(\boldsymbol{\xi}, t) = Ie^{i\omega t}\delta(\rho' - \rho_0)\delta(z' - 1)\hat{\boldsymbol{\phi}}', \quad (55)$$

using cylindrical coordinates $(\rho, \phi, z)$. The fields are shown in figure 3 and 4 were found again using Mathematica's numerical integration package and plotting with arbitrary values.

## IV. Conclusion

In this paper we have combined the extension of distribution theory to vector distributions from reference [6] with the formalism of scalar Green's functions. We have applied this theory to Electrodynamics and have re-derived more generally several commonly used formulae, as well as discovering several uniqueness conditions for vector potentials.

Future work involves deriving the fields in the lower half-spaces for conducting, dielectric, and paramagnetic materials, as well as extending the finite boundary method to vectors in order to deal with inhomogeneities.

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