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ICE-SLOPE INTERACTION: TRANSITIONS IN FAILURE MODE

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ABSTRACT

A two-dimensional hydrodynamic ice-slope interaction problem is formulated in this paper. The solution of this elastohydrodynamic problem will give the desired ice force. The main objective is to determine the influence that ice sheet velocity, slope angle, and ice-structure friction have on the ice force. A transition in failure mode is often observed to occur at a certain velocity (named the transition velocity) from predominantly bending to predominantly crushing. In this paper, the solution is provided for relatively slow interactions, to illustrate the validity of the approach. The treatment is non-dimensional so that different types of ice (model ice, freshwater ice and sea ice) can easily be related.

INTRODUCTION

The first objective of this paper is to formulate an approach which yields the ice force $\bar{N}$ as the solution of the problem solved. Second, assuming the acceleration or velocity of the ice sheet is explicitly included in the problem statement, it should be possible to ascertain the functional dependencies of the failure strain, and breaking lengths, on the typical input parameters (characteristic length, ℓ; flexural strength, σ_f; ice thickness, h; ice velocity, v; ice-structure friction, f). For the sake of simplicity, the formulation and analysis are two-dimensional; for this reason, the influence of the structure width cannot be ascertained (Timco, 1984; Frederking and Timco, 1985).

By solving one or two fully elastohydrodynamic problems, it is hoped that an effective approximate technique can be devised to include velocity-hydrodynamic effects that will thereby facilitate the accurate solution of the more complicated three-dimensional problems.

In this paper two closely related but necessarily idealized problems (see Figures 1a and 1b) are examined to gain some insight into the reasons for an observed transition in failure mode. A non-dimensional formulation is presented. For this reason, the analysis is applicable not just to model ice but also to freshwater (river and lake) ice as well as sea ice.

ICE-SLOPE INTERACTIONS

Haynes *et al.* (1983) conducted a total of 64 tests in the 33.5-m long, 9.15-m wide and 2.4-m deep refrigerated test basin at the U.S. Army Cold Regions Research and Engineering Laboratory (USACRREL). Using urea model ice, 22 ice sheets were pushed against a model sloping structure with varying slope angles. During 31 of these tests, the velocity of the ice sheet was increased at a constant rate from 0 to 10 cm/s over a period of 90 seconds. The horizontal force exerted by the ice on the structure was measured. A transition in the failure mode was observed to occur at a certain velocity (and named the transition velocity) from predominantly bending to predominantly crushing. For the given test duration studied (which involved 4.5-m of travel) this transition occurred for relatively high slope angles only.

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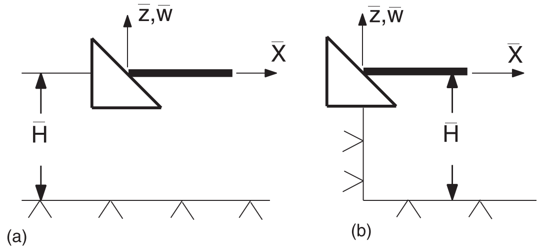

Figure 1. A semi-infinite ice cover impacting a sloping structure: fluid movement beneath the structure is (a) unrestricted and (b) restricted.

PROBLEM DESCRIPTION

A semi-infinite ice sheet of thickness h and width B, floating in water of constant depth $\bar{H}$, is being forced against a sloping structure, as portrayed in Figure 1. For time $t \leq 0$, the ice sheet is unloaded and motionless with the weight of the sheet being balanced by buoyancy. The associated immersed depth of the ice is given by $\rho_w h_0 = \rho_i h$, with ρ_w and ρ_i being the densities of the ice and water, respectively. The vertical deflection $\bar{w}(\bar{x},t)$ of the ice sheet for $t > 0$ is assumed to occur about the equilibrium configuration occupied by the ice sheet for $t \leq 0$. Typically, the influence of the hydrodynamic reaction on the subsequent deflexions is significant, especially for early times. (Dempsey & Zhao 1993; Zhao & Dempsey 1996). The deflexions occurring prior to breakup are generally much less than the ice sheet thickness. Partial emergence of the ice sheet prior to fracture is therefore not considered in the following analysis. Since a significant portion of the ice sheet thickness is submerged prior to uplift by the slope, a large deformation beam theory would be required to model this partial emergence. In deriving the boundary conditions governing the interaction between the ice sheet and the structure, the ice sheet thickness is taken into account. However, for the remainder of the formulation and analysis, the ice sheet is assumed to be of zero thickness, occupying the semi-infinite region $0 \leq \bar{x} < \infty$, $-B/2 \leq \bar{y} \leq B/2$, $\bar{z} = 0$.

There are two types of ice sheet-sloping structure interactions to be included in this paper. The first corresponds to unrestricted flow of the water around the structure, in which case the water will be assumed to occupy the region defined by $-\infty < \bar{x} < \infty$, $-B/2 \leq \bar{y} \leq B/2$, $-\bar{H} < \bar{z} < 0$ (see Figure 1a). Possible physical settings for this first type include river ice moving against a bridge pier, sea ice being driven against the sloping sides of an offshore lighthouse or oil recovery platform, and a model ice sheet (in a test basin) being driven against a sloping but narrow (in width) structure. The second type corresponds to certain models basin studies in which the sloping structure is as wide as the ice sheet being encountered (Yean *et al.* 1981; Hopkins 1995). In this latter case, only the water directly below the ice sheet will be taken into account, occupying the region $0 < \bar{x} < \infty$, $-B/2 \leq \bar{y} \leq B/2$, $-\bar{H} < \bar{z} < 0$ (see Figure 1b).

The bottom-surface pressure due to the acceleration of the water is treated by assuming that the motion of the water is governed by two-dimensional potential theory (Kheisin 1967; Nevel 1970; Dempsey & Zhao 1993; Fox & Squire 1994). The water velocity vector $\mathbf{v}$ is expressed as the gradient of the velocity potential $\bar{\phi}(\bar{x},\bar{z},t)$. Assuming that the water is incompressible and inviscid, conservation of water mass requires that

$$\nabla^2\bar{\phi} = \frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\phi}}{\partial\bar{z}^2} = 0, \qquad -\infty < \bar{x} < \infty, \ -\bar{H} < \bar{z} < 0. \quad (1)$$

At the deformed ice-water interface, the vertical velocity of the fluid is equal to the vertical velocity of the ice. Given that the depth $\bar{H}$ of the fluid is constant, the vertical velocity at $\bar{z} = -\bar{H}$ is zero. The horizontal velocity at $\bar{x} = 0$ in Figure 1b is zero. Physically, as $\bar{x}$ approaches infinity, the water velocity is zero; correspondingly, $\bar{\phi}$ tends to a constant value which is arbitrarily set to zero. The pressure $\bar{p}_i$ in (2) is given by Bernoulli's equation, the term proportional to the square of the vertical velocity being omitted; the influence of this term is assumed to be insignificant, although this assumption will be checked by comparing the energies of relevant quantities. The motion of the floating sheet is modeled using Euler-Bernoulli beam theory. Thus, on the ice-covered portion of the boundary ($0 < \bar{x} < \infty$, $\bar{z} = 0$),

$$c_a\rho_i h\frac{\partial^2\bar{w}}{\partial t^2} + E'\hat{I}\frac{\partial^4\bar{w}}{\partial\bar{x}^4} + \bar{N}\frac{\partial^2\bar{w}}{\partial\bar{x}^2} - \bar{p}_i(\bar{x},t) + \rho_i hg = \bar{q}(\bar{x},t),$$

$$\frac{\partial\bar{\phi}}{\partial\bar{z}} = \frac{\partial\bar{w}}{\partial t}, \qquad \bar{p}_i = -\rho_w g\bar{\eta} - c_h\rho_w\frac{\partial\bar{\phi}}{\partial t} + \rho_w g h_0, \quad (2)$$

where $E'\hat{I}$ is the flexural rigidity per unit width of the ice sheet. The notation E' is introduced to treat both plane stress and plane strain: E' equals E for plane stress and $E/(1-\nu^2)$ for plane strain, with with B equal to unity in the latter case. $\hat{I}$ equals $h^3/12$. The vertical deflection of the ice sheet from its unloaded equilibrium position is denoted by $\bar{w}(\bar{x},t)$ while $\bar{q}(\bar{x},t)$ represents the external loading (pressure) applied to the top surface of the ice sheet, $\bar{p}_i$ is the fluid-ice interface pressure exerted by the hydrodynamic reaction on the bottom surface of the ice sheet, $\bar{N}(t)$ is the in-plane compressive ice force per unit width generated by the ice sheet movement against the sloping structure. In $(2)_3$ $\bar{\eta}(\bar{x},t)$ is the displacement at the ice-water interface; since the ice sheet will fracture well before cavitation is a possibility, the latter displacement is matched with the displacement of the ice sheet

$$\bar{\eta} = \bar{w}, \qquad 0 < \bar{x} < \infty, \ \bar{z} = 0. \quad (3)$$

The coefficients c_a and c_h are introduced in (2) to enable this problem description to include both the fully coupled elastohydrodynamic model and a uncoupled, somewhat *ad hoc* added mass model. For the elastohydrodynamic model $c_a = 1$ and $c_h = 1$ in $(2)_1$ and $(2)_3$, respectively; for the particular added mass model considered here (Sørensen 1978; Sodhi 1987) $c_h = 0$ but c_a is specified *a priori*, the fluid interaction is not included, leaving $\bar{p}_\text{i} = -\rho_\text{w} g\bar{w}$. The effectiveness of added mass models of this type (the type in which the mass density of the ice sheet is simply multiplied by some *ad hoc* number) versus the fully coupled elastohydrodynamic analysis, for the problem being considered in this paper, will be examined.

On the lower boundary $\bar{z} = -\bar{H}$, in Figure 1a (for $-\infty < \bar{x} < \infty$) or Figure 1b (for $0 < \bar{x} < \infty$), the potential satisfies $\partial\bar{\phi}/\partial\bar{z} = 0$ while on the boundary $\bar{x} = 0$, $-\bar{H} < \bar{z} < 0$ in Figure 1b, $\partial\bar{\phi}/\partial\bar{x} = 0$. The free surface boundary in Figure 1a requires that (Fox & Squire 1994)

$$\frac{\partial^2\bar{\phi}}{\partial t^2} + g\frac{\partial\bar{\phi}}{\partial\bar{z}} = 0, \qquad -\infty < \bar{x} < 0,\ \ \bar{z} = 0. \tag{4}$$

At any section in this ice sheet, the bending moment per unit width $\bar{M}(\bar{x},t)$ and the vertical shear force per unit width $\bar{Q}(\bar{x},t)$ are given (noting the sign convention portrayed in Figure 2) by

$$\bar{M} = E'\hat{I}\frac{\partial^2\bar{w}}{\partial\bar{x}^2}, \qquad \bar{Q} = \frac{\partial\bar{M}}{\partial\bar{x}} + \bar{N}\frac{\partial\bar{w}}{\partial\bar{x}}. \tag{5}$$

With the slope angle α as indicated in Figure 2, and with the coefficient of friction being denoted by $\mu = \tan\vartheta$, it is evident that

$$\bar{M}(0,t) = -f\frac{h}{2}\{R_x(t) + R_z(t)\frac{\partial\bar{w}}{\partial\bar{x}}(0,t)\}, \qquad R_x(t) = \bar{N}(t),$$

$$\bar{Q}(0,t) = R_z(t), \qquad R_z(t) = \bar{N}(t)/\tan(\alpha+\vartheta), \tag{6}$$

in which R_x and R_z are the horizontal and vertical components of the time-varying force applied to the ice sheet at $\bar{x} = 0$; f is a factor to account for crushing at $\bar{x} = 0$ (for no crushing, $f = 1$). This 'contact factor' f may range between $f = +1$ ($\bar{z} = -h/2$) and $f = 1$ ($\bar{z} = h/2$). Luk (1987) introduced a similar contact factor. Especially for columnar freshwater or sea ice, this range on f seems rather unrealistic. While crushing is expected to initiate at the right-angle corner that first meets the slope, the high compressive strength of ice under combined loading conditions means that a relatively small crushed area generates a significant uplift force. Moreover, given the relatively brittle behavior of ice, the magnitude of the vertical displacement is expected to be small relative to the ice sheet thickness. The influence of the value of f will be investigated.

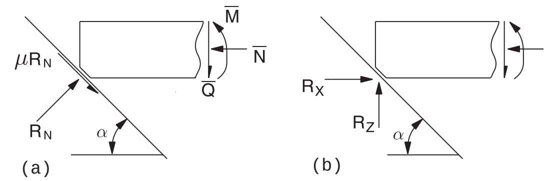

Figure 2. Contact forces and sign conventions.

Note that for the critical slope angle $\alpha = \alpha_\text{cr} = \pi/2 - \vartheta$, the vertical end force $R_z = 0$. The equations in (6) hold for $\alpha \le \alpha_\text{cr}$. For slope angles greater than or equal to α_cr the ice sheet will not move up the slope (unless the under-ice fluid motion causes an upward motion of the ice sheet end, overriding the frictional restraint), and the ice will fail either by buckling and/or crushing. Note that the elastostatic buckling load for a semi-infinite ice sheet is given by $\bar{N}_\text{b} = (\rho_\text{w} g E'\hat{I})^{1/2}$ (Hetényi 1946). For $\alpha < \alpha_\text{cr}$ the ice will move up the slope and fail due to bending, although at high values of α failure has been observed to be caused by a combination of buckling and bending (Haynes *et al.* 1983).

The time-varying in-plane force $\bar{N}(t)$ is given by $(5)_2$ and $(6)_2$ as

$$\bar{N}(t) = E'\hat{I}\frac{\partial^3\bar{w}}{\partial\bar{x}^3}(0,t)\Big/\left\{\frac{1}{\tan(\alpha+\vartheta)} - \frac{\partial\bar{w}}{\partial\bar{x}}(0,t)\right\}. \tag{7}$$

In the experiments conducted by Haynes *et al.* (1983), the ice sheet is pushed against the sloping structure at a constant acceleration a_0 from zero initial velocity; the end of the ice sheet is therefore forced to move up as

$$\bar{w}(0,t) = a_0 t^2 \tan\alpha\,/2, \tag{8}$$

subject to the end moment specified via (6),

$$\bar{M}(0,t) = -f\frac{h}{2}\bar{N}(t)\left\{1 + \frac{\partial\bar{w}}{\partial\bar{x}}(0,t)/\tan(\alpha+\vartheta)\right\}. \tag{9}$$

In the above experiments, the ice sheet is actually 10 to 20 meters long; the far-field acceleration is uniform and controlled. In other words, the acceleration of the ice sheet is controlled, not the in-plane ice-force, $\bar{N}(t)$.

Given that the compressive strength $\bar{\sigma}_c$ of ice is generally much greater than the tensile or flexural strength (as is typical of quasi- brittle materials), any maximum in the ice force $\bar{N}(\bar{x},t)$ must be associated with a peak in the axial stress $\bar{\sigma}(\bar{x},t)$ when the amplitude of this peak matches the flexural strength $\bar{\sigma}_f$ of the ice. Correspondingly, the failure criterion for the axial stress at any point $(\bar{x},\bar{z})$ is given by

$$-\bar{\sigma}_c \le \bar{\sigma}(\bar{x},\bar{z},t) = -\frac{\bar{N}}{h} - \frac{\bar{M}(\bar{x},t)\,\bar{z}}{\hat{I}} \le \bar{\sigma}_f, \qquad 0<\bar{x}<\infty, \bar{z}=0 \tag{10}$$

NON-DIMENSIONAL NOTATION

It is useful to introduce non-dimensional quantities and co-ordinates, as follows. Let

$$x = \bar{x}/\ell, \qquad z = \bar{z}/\ell, \qquad w = \bar{w}/h,$$

$$\phi = \bar{\phi}/h\sqrt{g\ell}, \qquad \tau = t/\sqrt{\ell/g}, \qquad H = \bar{H}/\ell,$$

$$N = \bar{N}/\bar{N}_b, \qquad M = \bar{M}/\bar{N}_b h, \qquad Q = (\bar{Q}/\bar{N}_b)(\ell/h),$$

$$\sigma = \bar{\sigma}h/\bar{N}_b, \qquad \sigma_c = \bar{\sigma}_c h/\bar{N}_b, \qquad \sigma_f = \bar{\sigma}_f h/\bar{N}_b, \tag{11}$$

in which

$$\bar{N}_b = (\rho_w g E'\hat{I})^{1/2} = \rho_w g \ell^2, \qquad \text{and} \qquad \ell = \left(\frac{E'\hat{I}}{\rho_w g}\right)^{1/4}. \tag{12}$$

As has already been noted, $\bar{N}_b$ in (12) is the ice force per unit width at a the semi-infinite floating ice sheet will buckle, if the end at $x = 0$ is loaded solely by a vertical concentrated force (Hetényi, 1946); ℓ is called the characteristic length of the ice sheet (Sodhi *et al.* 1982).

RELATIVELY SLOW INTERACTIONS

If the ice sheet motion were to be slow enough to substantially diminish the role of the fluid, the problem reduces to the solution of

$$\frac{d^4w}{dx^4} + N\,\frac{d^2w}{dx^2} + w = 0, \qquad 0<x<\infty,\ z=0, \tag{13}$$

in which

$$N = \frac{d^3w}{dx^3}(0) \Bigg/ \left\{ \frac{\ell}{h}\,\frac{1}{\tan(\alpha+\vartheta)} - \frac{dw}{dx}(0) \right\}, \tag{14}$$

subject to $w(x\to\infty)=0$, $w'(x\to\infty)=0$, and

$$w(0) = \frac{a_0}{2g}\tan\alpha\frac{\ell}{h}\tau^2,$$

$$\frac{d^2w}{dx^2}(0) = -f\frac{N}{2}\left\{1 + \frac{h}{\ell}\frac{1}{\tan(\alpha+\vartheta)}\frac{dw}{dx}(0)\right\}. \tag{15}$$

The solution to (13), (14) and $(15)_2$ is given by

$$w(x) = \exp(-\lambda x)(A\cos\kappa x + B\sin\kappa x), \tag{16}$$

in which

$$\lambda = (2-N)^{1/2}/2, \quad \kappa = (2+N)^{1/2}/2, \tag{17}$$

$$A = \left(\frac{2\ell}{hT}\lambda N - \frac{f}{2}N - \frac{f}{2T^2}N^2\right)/D,$$

$$B = \left(\frac{f}{2}\lambda N - \frac{\ell}{2hT}N^2 - \frac{f}{2T^2}\lambda N^2\right)/\kappa D,$$

$$D = 1 - N - \frac{fh}{\ell T}\lambda N, \quad T = \tan(\alpha+\vartheta). \tag{18}$$

By examining the equation $D = 0$, it is found that an accurate approximation to the normalized buckling load is provided by

$$N_{cr} = \frac{2}{2+\xi}, \quad \text{with} \quad \xi = \frac{fh}{\ell T}. \tag{19}$$

The movement (uplift) of the ice sheet is specified by $(15)_1$; this condition implies that

$$N\frac{4\lambda-\xi T^2-\xi N}{1-N-\xi\lambda N}=\frac{a_0 T}{\ell}\,t^2\tan\alpha. \tag{20}$$

This equation may be solved to provide N. For later use note that

$$\frac{d\bar{w}}{d\bar{x}}(0)=-N\big(1-\xi T^2\lambda\big)/DT. \tag{21}$$

Proceeding from equation (10) (with the time dependence removed), it is evident that the maximum stress will occur for $w'''(x)=0$; this condition gives the predicted breaking distance x_{b} to be

$$x_{\mathrm{b}}=\frac{1}{\kappa}\arctan\left\{\frac{4\kappa(1-\xi\eta\lambda N)}{4\lambda-2\xi T^2+\xi\eta N^2}\right\},\quad \text{with}\quad \eta=\sec^2(\alpha+\vartheta). \tag{22}$$

HYDRODYNAMICS

Valanto (1992) states: ‘The first force peak in the test with presawn ice sheet is mainly caused by inertia related to the initial acceleration of the floating ice sheet and the surrounding water. In the test with unbroken ice, work is also done to produce the flexural failure in the ice sheet. This is a dynamic phenomenon and the difference between the initial force peaks of unbroken and presawn ice peaks increase with increasing speed of advance.’ Valanto (1989, 1992) constructed idealized 2D icebreaking experiments, and found that the maximum ice force increased steadily, while the length of the broken slab decreased steadily, with increase in velocity of the icebreaking model. The elastostatic solution for the piece lengths given in (22) does not predict this behavior. The dynamic response due to edge forcing has been studied previously by Sørensen (1978) and Sodhi (1987). In these studies each author included the hydrodynamic factors by an added mass factor. Dempsey and Zhao (1993), Fox (1993) and Zhao and Dempsey (1996) have illustrated that such an approach is not realistic. Because of the complexity associated with the hydrodynamics, the 3D problems solved to date do not accurately include the influence of ice velocity.

Valuable information remains to be harvested, should it become feasible to solve the 3D problems of interest. For instance, suppose an accurate scaling could be ascertained relating the ice velocity and structural diameter.

CONCLUSIONS

A two-dimensional hydrodynamic ice-slope interaction problem has been formulated. The solution of this elastohydrodynamic problem will give the desired ice force. An analytical solution has been provided for the case of relatively slow interactions. The importance of truly hydrodynamic solutions was emphasized.

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