Physical Acoustics – Models and Solutions

(Molecules to Modes and back)

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Abstract

Mathematical modelling in the study of physical acoustics has proved to be extremely accurate and useful in some situations (within small enclosed spaces) and completely useless in others (large open spaces). We derive the usual mathematical model for sound propagation from simple principles so that its range of validity can be seen. Some analytical and numerical methods of solution are briefly discussed.

1 Introduction

This paper, and talk, address the issue of mathematically modelling the propagation of sound and how we can use the models to calculate features of a sound field – without doing experiments. Modelling usually start with what we know about the system being modelled. For our starting point I'll use the fairly common observation that sound is produced when something vibrates (a wall, a larynx) and pushes air backwards and forwards. The disturbance created in the air propagates and we hear sound.

1.1 Molecules of Air

We know that air is made up of molecules that are jigling around. The air interacts with the wall through collisions of individual molecules with the wall – a bit like peas bouncing on a roof except on a smaller and faster scale.

We might first think of building a mathematical model by representing the position and velocity of each air molecule and then simply follow their motion to find how sound propagates. But since there is about 2.5×10^{25} molecules per cubic metre of air[1] there is little future in this approach.

The huge number of molecules is a mixed blessing. While it is too large to work with it does mean that we can represent the air by *average* properties to a very good accuracy because we know that sound only occurs as a cooperative activity between very many molecules. For the purpose of the averaging process we will consider a cube of air, 1μm on each side, which has five rigid sides and one that can act as a piston. How did I know the number of molecules in a cubic metre? Because the pressure P, volume V, number of molecules n and absolute temperature T are related by

$$PV = nkT$$

where $k = 1,381 \times 10^{-23}$J/K is Boltzman's constant. So at atmospheric pressure (1.013×10^5 N/m^2 and room temperature (293 K) that puts a managable 2.5×10^7 molecules in the $(1\mu\text{m})^3$ cube of air. Certainly enough to average over. And since each molecule is moving with (rms) speed of 5×10^6m/s (!!), each molecule will bang against each wall of the box a million times each micro-second, so the averaging in time of the individual collisions is a very good approximation.

[1] and less than 10^{16} bytes of memory exist

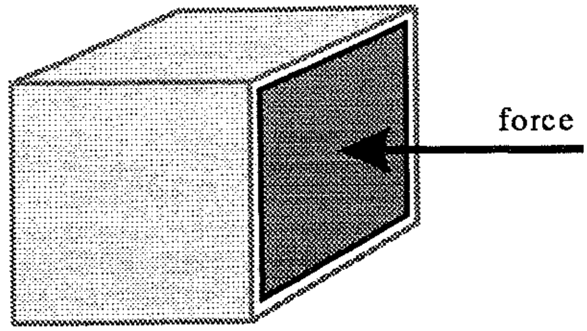

Figure 1: $(1\mu\mathrm{m})^3$ of air with force applied to one face

The relationship between pressure applied on one face to the volume can be found from the previous equation – at least if the pressure is applied slowly. In that case the molecules have time to stay at thermal equilibrium and so the temperature remains constant[2] and we find

$$Pp + Vv = 0 \qquad \text{or} \qquad p = -\frac{P}{V}v$$

where p and v are the changes in the mean values of pressure P and volume V, respectively. If however, the pressure is applied rapidly so that no heat has time to flow[3] we find instead that

$$p = -1.4\frac{P}{V}v. \tag{1}$$

As the pressure is applied the box of air will move according to the difference of pressures on opposite faces. For example the difference in pressure in the x direction is $\frac{\partial p}{\partial x} \times 1\mu\mathrm{m}$ and so the nett force in the x direction is $\frac{\partial p}{\partial x} \times 1\mu\mathrm{m} \times 1\mu\mathrm{m}^2$. It is convenient to write

$$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$$

so we can write the total nett force as

$$\text{nett force} = \nabla p \times V.$$

Newton's second law, $F = ma$, tells us that if $\mathbf{q}$ is the average velocity of the molecules in the box then the average acceleration is

$$\rho\frac{\partial \mathbf{q}}{\partial t} = -\nabla p \tag{2}$$

where ρ is the density of air.

The box will also be squashed a little by the extra pressure. We can relate the change in volume v to the velocities $\mathbf{q}$ as follows. The difference of velocities of opposite faces in the x direction is $\frac{\partial q_x}{\partial x} \times 1\mu\mathrm{m}$ and so the difference of distances travelled in some short time, say $1\mu\mathrm{s}$, is $\frac{\partial q_x}{\partial x} \times 1\mu\mathrm{m} \times 1\mu\mathrm{s}$. The resulting change in volume is $\frac{\partial q_x}{\partial x} \times 1\mu\mathrm{m} \times 1\mu\mathrm{s} \times 1\mu\mathrm{m}^2$. The total change in volume, taking the three coordinate directions into account is $\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) V \times 1\mu\mathrm{s}$. The term $\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right)$ is concisely written $\nabla \cdot \mathbf{q}$ (notice the dot) and since the change in volume in $1\mu\mathrm{s}$ equals $\frac{\partial v}{\partial t} \times 1\mu\mathrm{s}$ we have

$$\frac{\partial v}{\partial t} = \nabla \cdot \mathbf{q}. \tag{3}$$

[2] this cange is isothermal

[3] this change is adiabatic

Equations (1), (2), and (3) may be combined to eliminate the variables other than p – which is the sound pressure – to give

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

where $c = \sqrt{1.4P/\rho}$ and $\nabla^2 p = \nabla \cdot (\nabla p)$ is the Laplacian. It should be noted that the Laplacian is unchanged in form when the coordinates are rotated.

1.2 Particular Solutions

Now that we have a mathematical model for the way the acoustic pressure propagates, we can manipulate the model to find features of the sound field, or every detail of the sound field, in spaces and with acoustics sources of interest.

We typically use the model to predict acoustic propagation in two ways: The first, in the spirit of our derivation, is to *simulate* the acoustic pressure as it develops in time and space away from some source. For example, if we know the sound pressure at some initial instant and we know the sound pressure generated at any sources then we can follow the development of of the pressure in time at all points in space. Usually we would start with silence, i.e. $p = 0$ at $t = 0$, turn on the acoustic source(s) and at each instant in time use the spatial dependence to evaluate the left-hand side of the wave equation which tells us how the pressure changes in time – since that is the right-hand side. The second less obvious use is to verify that a given function $p(\mathbf{x}, t)$, depending on spatial coordinates $\mathbf{x} = (x, y, z)$ and on time t, satisfies the wave equation and hence is an allowable sound pressure field. This approach is often used in simple cases where it is possible to make an educated guess at the sound pressure field and the guess is verified by seeing that it satisfies the wave equation.

The educated guess required for the second usage is made easier by there being a wide class of functions that satisfy the wave equation; If $g(u)$ is *any* function then

$$p((x, y, z), t) = g(x - ct)$$

solves the wave equation[4]. Note that $p((x, y, z), t)$ represents a disturbance propagating in the positive-x direction at speed c as shown in Figure 1. As mentioned earlier, the part of the

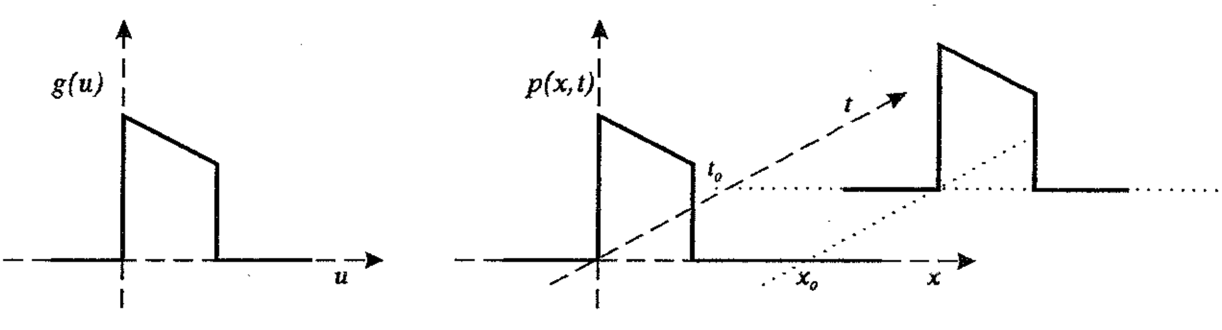

Figure 2: The sound pressure $p((x, y, z), t) = g(x - ct)$ propagates to the right, so at time t_0 the distrubance has moved by $x_0 = c/t_0$.

wave equation depending on the space variables is unchanged by rotation of the coordinates.

[4]It is easy to check that $p((x, y, z), t) = g(x - ct)$ solves the wave equation since $\frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 g}{\partial u^2}(x - ct)$ while $\frac{\partial^2 p}{\partial y^2} = 0 = \frac{\partial^2 p}{\partial y^2}$ and $\frac{\partial^2 p}{\partial t^2} = (-c)^2 \frac{\partial^2 g}{\partial u^2}(x - ct)$ and so $\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$ as desired.

So there is nothing special about the x-axis and we could have set $p((x,y,z),t) = g(y - ct)$ or $p((x,y,z),t) = g(z - ct)$ to form disturbances in the positive-y or positive-z directions, respectively.

More generally it is not too difficult to show that

$$p((x,y,z),t) = g(k_x x + k_y y + k_z z - \sqrt{k_x^2 + k_y^2 + k_z^2}ct).$$

also satisfies the wave equation and so gives a possible sound pressure field. In this case $p((x,y,z),t)$ represents a disturbance propagating in the direction of the vector (k_x, k_y, k_z) with speed c. The most important example from this class of pressure fields occurs when the underlying function is a sine wave and we find the solution

$$p((x,y,z),t) = \sin\left(k_x x + k_y y + k_z z - \sqrt{k_x^2 + k_y^2 + k_z^2}ct\right).$$

It is conventional (and brief) to write

$$\mathbf{k} = (k_x, k_y, k_z) \qquad k = \sqrt{k_x^2 + k_y^2 + k_z^2} \qquad \omega = kc$$

allowing the shorthand form

$$p(\mathbf{x}, t) = \sin(\mathbf{k} \cdot \mathbf{x} - \omega t).$$

This sound pressure is a plane wave of single frequency $f = \frac{\omega}{2\pi}$ and, equivalently, of wavelength $\lambda = \frac{2\pi}{k}$ propagating in the direction $\mathbf{k}$. Note that the *length* of the vector $\mathbf{k}$ is proportional to the frequency while the *direction* of $\mathbf{k}$ equals the direction of propagation.

Another important solution is given by the more esoteric function

$$m(\mathbf{x}, t) = \frac{\sin(k\|\mathbf{x}\| - \omega t)}{4\pi\|\mathbf{x}\|}$$

which is an outward travelling spherical wave such as radiates from a small single-frequency source located at $(x,y,z) = (0,0,0)$. Because of its spherical symmetry[5], this is the field resulting from a harmonicly oscillating monopole located at the origin.

The previous solution can be further modified to find other, important solutions. For example the gradient along the z-direction of the monopole sound field is

$$d_z(\mathbf{x}, t) = \frac{z}{4\pi}\left(\frac{k\cos(k\|\mathbf{x}\| - \omega t)}{\|\mathbf{x}\|^2} - \frac{\sin(k\|\mathbf{x}\| - \omega t)}{\|\mathbf{x}\|^3}\right)$$

and this also is an allowable sound field because it satisfies the wave equation[6] This solution corresponds to the sound field produced by a dipole source such as an unbaffled loudspeaker driven at a single low frequency.

[5] The sperical symmetry can be seen because the spatial dependence of p is determined by $\|\mathbf{x}\|$ alone which is the distance from the origin.

[6] This follows because the action of taking the gradient in the z-direction (any direction) *commutes* with the operations ∇^2 and $\frac{\partial^2}{\partial t^2}$ in the wave equation.

1.3 Does it work?

The sound fields found in the last section are all ones that simply propagate away from the source with no reflections occuring. So we might hope to apply these model fields to outdoor propagation problems where sound can travel large distances without meeting an obstacle. However, comparisons of measurements made over distances of about 1km with the theory are very disapointing. In fact, it is difficult to find such cases where the there is any merit in our model.

For example, from my house in Onehunga I can see the motorway one kilometre away approaching the harbour bridge[7] and am able to watch the rush-hour traffic while I eat breakfast. Some mornings I can also hear the traffic noise, it can sometimes sound as though the cars are only a hundred metres away, and yet other mornings the clearly visible cars are completely inaudible to me. Being of a scientific disposition I have noticed that the difference in audibility, among mornings when the air is still, correlates with atmospheric conditions and particularly with air temperature. This kind of discrepency allows us to improve the model by examining the way temperature entered the model. In our derivation the temperature was assumed to be constant and the resulting value for the speed of sound, c, was given by the constant expression $c = \sqrt{1.4P_0/\rho_0}$. It is clear that this expression will not be constant if temperature varies since at a given pressure hotter air is less dense. Including this effect we find the speed of sound is given by

$$c = 20.06\sqrt{273 + T} \qquad \text{(metres per second)}$$

when T is given in degrees Celsius. Thus the speed of sound propagation increases with air temperature. Since wave-fronts bend towards regions of lower speed we can now give a qualitative explanation of the motorway noise observation as depicted in Figure 2. If the air temperature

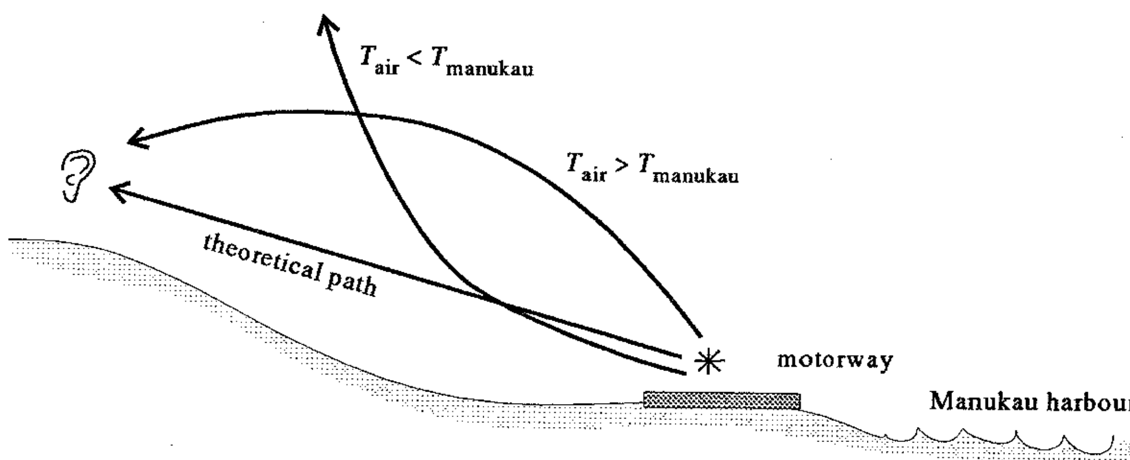

Figure 3: Paths for ray-like propagation of motorway noise in the presence of a vertical gradient of temperature.

increases with height, the motorway noise is bent downwards and I hear sound that is unimpeeded by trees and other objects at ground level – hence the traffic sounds close. Conversely, when the air temperature decreases with height all the motorway noise is bent upwards and I am able to enjoy my breakfast in a region of sound shadow.

Interestingly, the improved model has allowed us to make a qualitative description of the phenomena but we are unable to make detailed statements because that requires detailed *a*

[7]Auckland actually has many harbour bridges

priori knowledge of the air temperatures. When the air is not still, scattering of the sound from turbulence and other inhomogineities causes a futher marked departure from the theory and renders useless even the qualitative model.

So the best we can hope for the model is to employ it in cases where the propagation is over modest distances and when any air flow is slow enough to avoid turbulences. In those cases, the model we derived turns out to be remarkably good.

1.4 Boundary Conditions

To model the sound propagation in rooms and other enclosed spaces where we hope the model will be valid, we need to include a model for the interaction of the air in the $(1\mu\text{m})^3$ cube with the enclosing surfaces.

For rigid walls the interaction is simple since the molecules bounce off the wall their average velocity into the wall is therefore zero. Thus the time derivative of the average velocity is also zero which implies, using equation (2), that the component of the gradient of pressure pointing into the surface is also zero. It is conventional to use the unit vector **n** at each point on the surface that is normal[8] to the surface. The statement that the normal velocity is zero is that $\mathbf{q} \cdot \mathbf{n} = 0$ and the equivalent statement using (2) is that $(\nabla p) \cdot \mathbf{n} = 0$. Since this equals the gradient of p in the direction of **n** we often write the condition at the surface as

$$\frac{\partial p}{\partial n} = 0 \qquad \text{at rigid surfaces.}$$

Most surfaces are not completely rigid and vibrate in response to the acoustic pressure incident upon them. In this case the velocity normal to the surface is not zero, but is related to the applied pressure by

$$Z_s \frac{\partial p}{\partial n} = p \qquad \text{at the surface}$$

where Z_s is the specific acoustic impedance of the surface.

More generally, the relationship between applied pressure and velocity is not local as implied by the previous equation. For a typical panel construction wall, acoustic pressure applied locally causes the entire panel to vibrate and hence generates velocity everywhere on the surface. Thus a more general boundary contact condition would relate pressure and velocity over the entire surface.

1.5 The usual model

The combination of the wave equation

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \qquad \text{at all points in space}$$

that describes propagation within the volume, the boundary condition, e.g.

$$\frac{\partial p}{\partial n} = 0 \qquad \text{on the surfaces}$$

that gives the behaiviour of the sound field at the surfaces that enclose the volume and the statement of the initial state of the sound pressure is called an *initial boundary-value problem*

[8] i.e. perpindicular

and it is usually statements of this form that are solved for particular spatial geometries and acoustic sources.

The two most important features of this set of equations is that they are *linear*, i.e., if two sound pressures satisfy the set of equations then so does their sum, and difference, and any multiple of them and also that they do not change with time. It is these mathematical properties that allow us to treat the single frequency solutions to the problem separately – and then sum the resulting solutions. When we make this simplification we formally write

$$p(\mathbf{x},t) = p(\mathbf{x})\sin(\omega t)$$

and now the space part satisfies Helmholtz' equation[9]

$$\nabla^2 p(\mathbf{x}) + k^2 p(\mathbf{x}) = 0.$$

2 Methods of Solution

One of the features of acoustics that make solving this equation a challenge is that the wavelength at typical frequencies is the smae general size as the objects in the sound field. In the extreme cases of very long or very short wavelength (compared to object sizes) the solutions may be found by approximations. When objects are much smaller than the wavelength the sound field is virtually unchanged by the presence of the object and a simple solution can be found. When the wavelength is much smaller than typical objects the sound field can be calculated as in geometric optics where objects cast sharp shadows. However, the centre of the audible range from 300 Hz to 3000Hz spans the wavelengths from 1 metre to 10 cm anf this is precisely the typical size of objects that we put in rooms. Consequently these simplifying assumptions are of little use and we are left with no option but to solve the equations exactly.

2.1 An Analytic Solution

A simple enclosed system that has an analytic solution is the sound pressure within a narrow rigid cylinder excited by an oscillating piston at one end. The sound pressure is

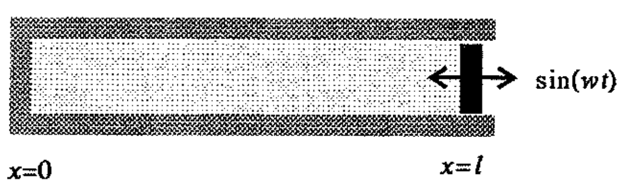

Figure 4: Narrow cylinder with a piston at one end

$$p(x,t) = -\frac{\cos(kx)\sin(\omega t)}{k\sin(kl)}$$

which can easily be verified by checking that the initial boundary value equations are satisfied. The spatial dependence can be seen to be unchanging in time – this is a standing wave and can be thought of as the sum of two wave-like solutions propagating left and right, respectively. These solutions are the modes of this system and the special cases where the denomenator $\sin(kl)$

[9]We have used the fact that $\frac{\partial^2 \sin(\omega t)}{\partial t^2} = -\omega^2 \sin(\omega t)$ and the relation $k^2 = (\omega/c)^2$.

equals zero are the *normal modes* of the system. The normal modes are solutions that exist in the absence of any forcing and correspond to the oscillation of the system after the sources have been stopped. In this case the normal modes occur at the discrete frequencies

$$f = \frac{n\pi c}{l} \qquad \text{where } n = 1, 2, 3, \cdots.$$

This geometry is a case where the solutions given by the model have been experimentally verified to great precision. A development of this system is routinely used for measuring the acoustic impedance of material placed at the closed end.

2.2 Boundary Element Method

The Boundary Integral Methods, like most efficient computational schemes, do not work with the wave equation or Helmholtz' equation directly but instead start with a mathematically equivalent form. Helmholtz' equation may be stated as a relationship between the sound pressure within the space and on the surfaces through an application of Green's theorem and identifying terms with the solution and the field due to a monopole source. The resulting expression is

$$p_{\text{source}}(\mathbf{x}) + \int_S \frac{\partial p}{\partial n}(\mathbf{x}')m(\mathbf{x}-\mathbf{x}') - p(\mathbf{x}')d_n(\mathbf{x}-\mathbf{x}')\,dS(\mathbf{x}') = \begin{cases} p(\mathbf{x}) & \mathbf{x} \text{ in space} \\ p(\mathbf{x})/2 & \mathbf{x} \text{ on surface} \end{cases}$$

The top line of this expression relates the pressure in a space $p(\mathbf{x})$ to the direct sound from any sources $p_{\text{source}}(\mathbf{x})$ and the pressure and its normal derivative on the enclosing surfaces S. The term in the integral hence gives the "scattered sound field" and it is interesting to note its structure. Note that the normal derivative of pressure on the surface propagates into the volume as a monopole while the pressure propagates as a dipole.

Once p and $\frac{\partial p}{\partial n}$ are known on the enclosing surface, the sound pressure throughout space can be calculated. Typically the surface conditions specify only *one* of these quantities; The Boundary Integral Methods consist of using that information and the second line of the expression, above, to give a second equation which together are solved for the two functions p and $\frac{\partial p}{\partial n}$ on the surfaces.

In the case of rigid surfaces, the boundary condition is that $\frac{\partial p}{\partial n} = 0$ leaving the equation

$$p_{\text{source}}(\mathbf{x}) - \int_S p(\mathbf{x}')d_n(\mathbf{x}-\mathbf{x}')\,dS(\mathbf{x}') = p(\mathbf{x})/2$$

which must hold at each $\mathbf{x}$ on the surface S and can be solved for $p(\mathbf{x})$ on S. The Boundary Element Method solves this integral equation by approximating it by an easily solvable matrix equation. The first step is to discretize the enclosing surface into small areas called elements. A cross-section through a discretization is shown in Figure 3.

The second step is to approximate the value of $p(\mathbf{x})$ on the surface by simple functions – e.g. constant over each element. If

$$p_i = \text{constant value of } p \text{ on element } i$$

$$f_i = \text{constant value of } p_{\text{source}} \text{ on element } i$$

and

$$K_{ij} = \int_{S_i} d_n(\mathbf{x}-\mathbf{x}')\,dS(\mathbf{x}')$$

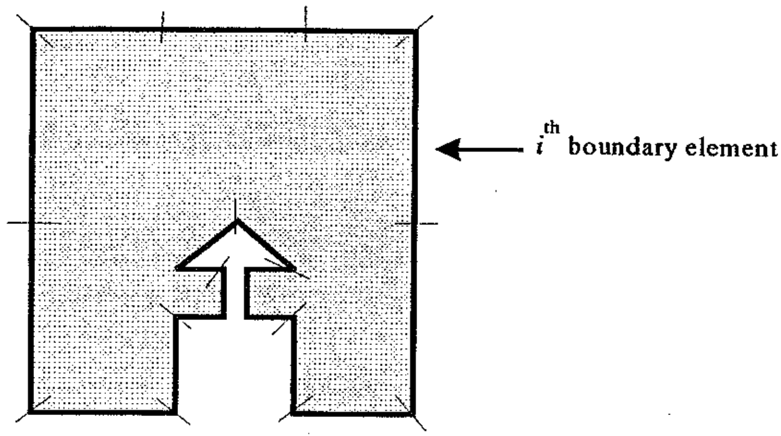

Figure 5: Discretization of the boundary of the volume

then the integral equation becomes

$$\left(K + \frac{1}{2}I\right) \text{bf p} = \text{bf f}$$

which can be solved for p using standard techniques built-in to good programming languages.

Finally, once both p and $\frac{\partial p}{\partial n}$ are known, $p(\mathbf{x})$ can be calculated at any point in the space.

The following two diagrams show the geometry and some results of calculations of the insertion loss for a motorway barrier.

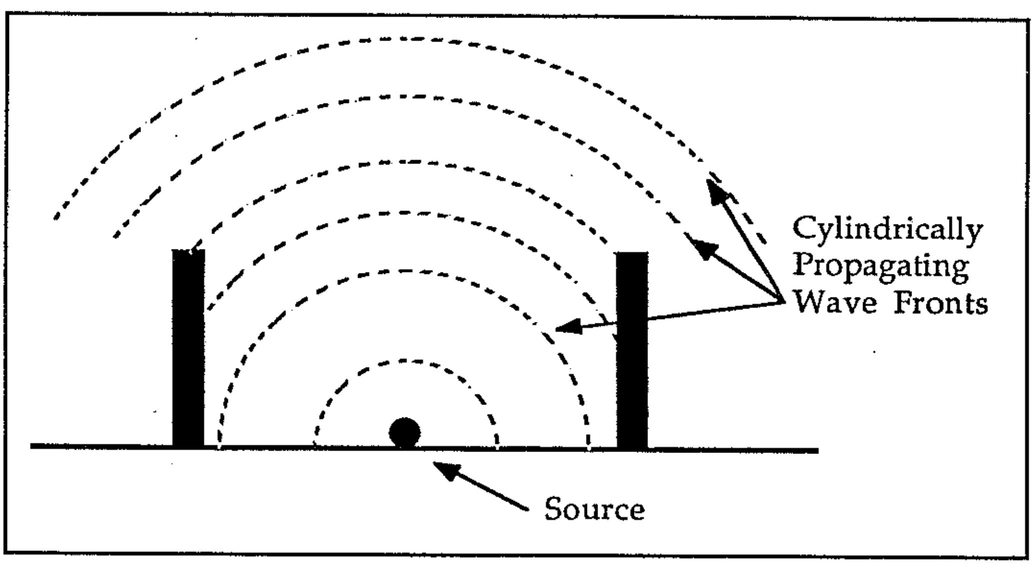

Figure 6: Cross-section through noise source (traffic) and sound barriers

2.3 Variational and Finite Element Methods

The wave equation is a local statement of the dynamics of air – we looked at a tiny volume to derive the equation. It has yet to cease amazing me that it is mathematically equivalent to a

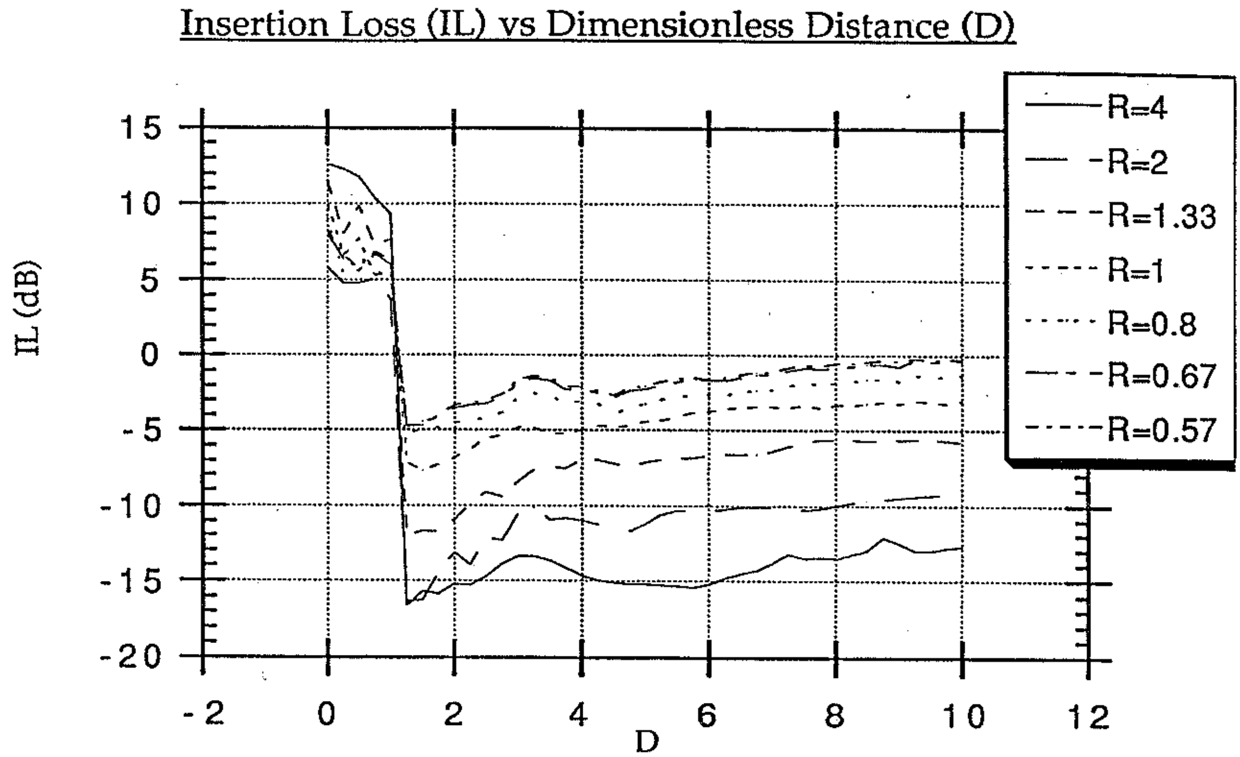

Figure 7: Insertion loss for various values of dimensionless distance D from the source and dimensionless height of the barrier R. The position of the barrier wall is clearly visible.

completely global statement of the dynamics known as Hamilton's principle. That equivalent statement is that the sound pressure develops in a way that minimises the total over time of the difference between the potential and kinetic energies of the air.

The Finite Element Method achieves the minimisation by first discretizing the volume into elements and then by approximating p by simple functions on each element – often linear functions. The minimum is calculated numerically over the finite number of elements, usually by solving the associated normal equations.

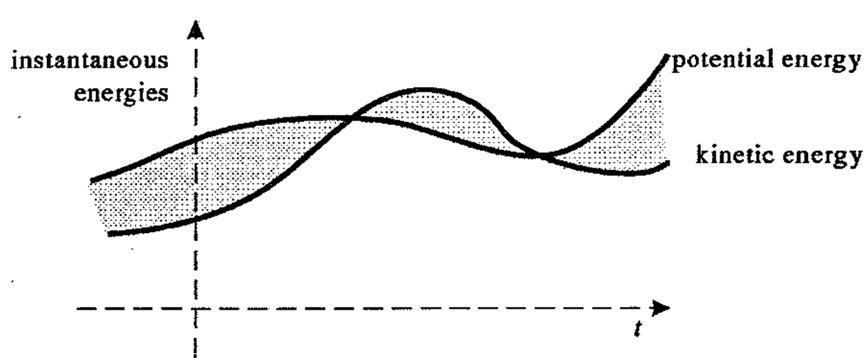

Figure 8: Shaded area is the total difference between potential and kinetic energy over time. Note that when kinetic > potential the area is considered negative.

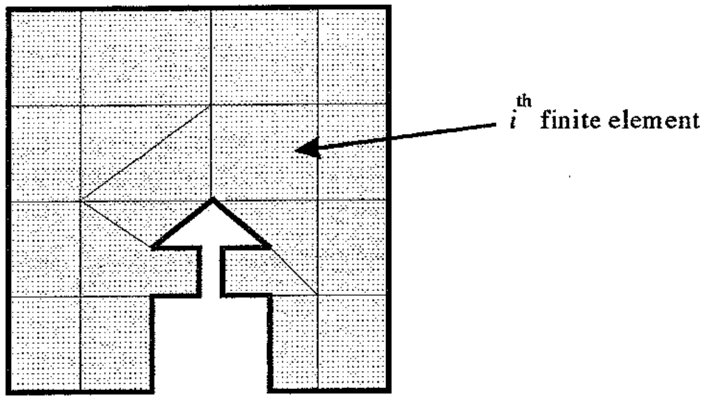

Figure 9: Discretization of the volume into elements

2.4 Finite Differences

In Finite Difference approximations the discretization is made by considering the solution value only at a finite number of points, called nodes, within the volume. The term $\nabla^2 p$ is approximated

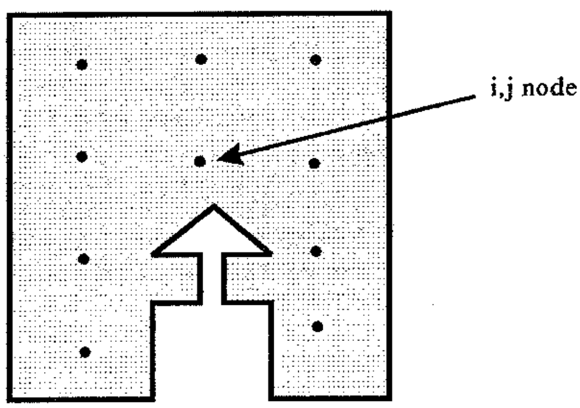

Figure 10: Discretization of the volume as nodes

by the finite-difference *molecule* acting on the grid of values. This leads to a matrix equation

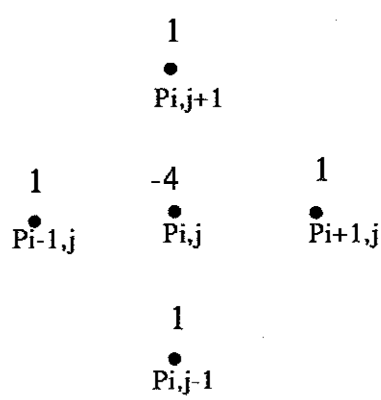

Figure 11: Finite difference molecule for Laplacian in two space dimensions.

for the unknown nadal values of p which can be solved giving the sound pressure directly.

3 Conclusions

The wave equation can be derived from simple ideas about the kinetics of the molecules in air; many of the details of dynamics of individual molecules are irrelevant. The resulting wave equation predicts that sound pressures will propagate with a constant speed – agreeing with observation over modest distances in enclosed volumes. Various methods of solution were discussed, the most widely applicable being the numerical Finite Element and Boundary Element methods.

4 References

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