CALCULATION OF WAVE-ICE INTERACTION USING THE WIENER-HOPF TECHNIQUE

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ABSTRACT. We present an analytic solution of a system of partial differential equations describing the oblique reflection and penetration of ocean waves into semi-infinite sea ice. The solution is obtained by the Wiener-Hopf technique by building on the earlier work of Evans and Davies. We simplify that derivation and extend it to derive efficient and stable numerical computation of solutions. Some examples of solutions are given.

35C10, 35Q35, 76B15

1. Introduction

The seasonal formation and breakup of sea ice has been of long-standing interest to New Zealand researchers [18, 11, 13], particularly those processes in the Southern Oceans around the coast of Antarctica. Significant features of the breakup process, as well as many details of the formation of the pack ice, are believed to be determined by the interaction between the sea ice and ocean waves [17, 16].

In this paper we address the problem of the interaction between ocean-going waves and a semi-infinite ice sheet, focussing on the calculation of the reflection of incident waves. This model idealizes the interaction between ocean waves and relatively straight-edged shore fast ice. The wave energy that is not reflected, i.e. that penetrates into the ice sheet, causes flexure of the ice with subsequent fatiguing and fracture [9]. We build on the solution method described by Evans & Davies [4] who derived analytical solutions of this wave-ice interaction problem using the Wiener-Hopf technique [14]. However, apparent complexity of their solution, and limited computing power at the time, prevented them from actually computating the solution. We will show that with a few modifications to the formulas and with knowledge of the roots of the dispersion equations for the free surface and the ice-covered region, it is straightforward to compute the solution using current computational tools.

The mathematical model that we analyze, outlined in the next section, treats the ice as a thin elastic plate resting on an inviscid irrotational fluid and originates from work by Greenhill in 1887 [12]. Solutions of this system have been available for some time with Fox and Squire in 1990 [8] giving the first computed solutions for regimes of geophysical interest, using a computational mode-matching technique. Prior to that work formal solutions had been derived, most notably by Evans and Davies [4] who used the Wiener-Hopf technique to derive an expansion of the solution with computation of solutions for asymptotic regimes. In 1999 Balmforth and Craster [1] extended that work by performing the required integral transforms using numerical quadrature to compute solutions for a model that incorporated a range of plate models including a thick plate. We note that they concluded, as was previously shown by Fox and Squire [10], that solutions are virtually unchanged from those using the thin plate model. In this paper we also complete the work by Evans and Davies, taking a more analytic route than Balmforth and Craster,

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by demonstrating that Evans and Davies' expansion may be directly computed. Our primary advance is to show that by focussing efforts into finding the roots of the dispersion equation, and by viewing functions as Mittag-Leffler expansions over those roots, the analytic manipulations are simplified while the computations required are relatively straightforward.

In sections 2 and 3 we give the model and a scaling to non-dimensional form. In sections 4 to 7 we derive the expansion of the solution using the Wiener-Hopf technique using a route that is a simplified form of that charted by Evans and Davies. Significant simplification is achieved by first establishing the general form of the expansion of the solution in section 4 thereby allowing us to easily establish the order-of-convergence results required later in the application of Liouville's Theorem in section. 6, in contrast to Evans and Davies lengthy calculations based directly on the system of differential equations. The numerical computations required are summarized in sections 7 to 9, with the latter section containing an example of computation of the reflection coefficient for a range of frequencies and incidence angles. Some details of the placement of the roots of the dispersion equations are presented in appendix A

2. Mathematical Model

At the wave periods of geophysical interest, roughly 2 seconds to 20 seconds, the ice sheet may be treated as a thin elastic plate and the sea water may be treated as an incompressible fluid [11]. The corresponding idealized model for wave-ice interaction consists of a semi-infinite thin plate of thickness h floating on the surface of an incompressible fluid of depth $\bar{H}$. We will solve for the response due to a plane wave obliquely incident on the ice edge from the open sea. Figure 1 shows a region of the water and ice sheet along with the associated water column and sea bottom for this idealized geometry. The coordinate system $(\bar{x}, \bar{y}, \bar{z})$ is also shown in relation to sea and ice sheet. The edge of the ice sheet is the line $\bar{z} = 0$, $\bar{y} = 0$, while the open sea is the region $\bar{y} < 0$ and the ice covers the surface in the region $\bar{y} > 0$. We consider a unit-amplitude plane wave propagating at angle θ with respect to the $\bar{y}$ axis.

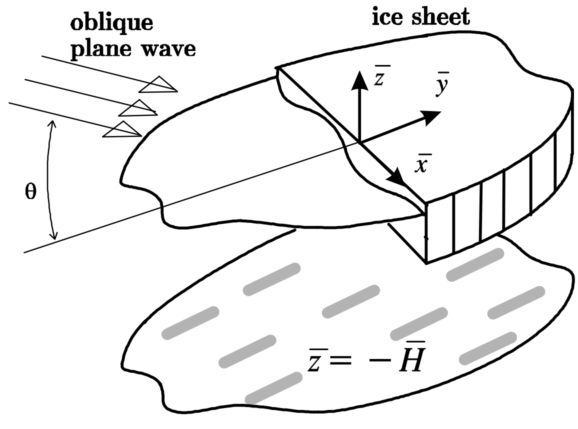

FIGURE 1. Schematic drawing of plane ocean wave obliquely incident at the edge of a sea ice sheet. The coordinate system used is located on the edge of the ice sheet and the sea floor is the plane $\bar{z} = -\bar{H}$.

We give a brief description of the classical boundary-value problem (BVP) that forms the resulting mathematical model [4]. Interested readers may find a derivation in Fox and Squire 1994 [11]. The motion of the fluid is parameterized by a velocity potential, $\bar{\phi}$, which satisfies Laplace's equation and a solid bottom condition

$$\nabla^2_{\bar{x},\bar{y},\bar{z}}\bar{\phi}\left(\bar{x},\bar{y},\bar{z}\right)=0,\ -\infty<\bar{x},\bar{y}<\infty,-\bar{H}<\bar{z}<0 \tag{1}$$

$$\left.\bar{\phi}_{\bar{z}}\right|_{\bar{z}=-\bar{H}}=0,\quad -\infty<\bar{x},\bar{y}<\infty. \tag{2}$$

Here $\nabla^2_{\bar{x},\bar{y},\bar{z}}$ denotes the Laplacian in the three space dimensions while $\bar{\phi}_{\bar{z}}$ denotes the partial derivative of $\bar{\phi}$ in the $\bar{z}$ direction. We will use analogous subscript notation for other partial derivatives. In the ice-covered region, $\bar{y}>0$, the vertical displacement $\bar{\eta}\left(\bar{x},\bar{y}\right)$ of the ice sheet satisfies the thin-plate equation

$$D\nabla^4_{\bar{x},\bar{y}}\bar{\eta}+m\left(\bar{\eta}_{\bar{t}\bar{t}}+g\right)=\bar{p}\quad\text{for}\quad -\infty<\bar{x}<\infty,0<\bar{y}<\infty \tag{3}$$

and is related to the potential by the linearized Bernoulli equation

$$\rho\bar{\phi}_{\bar{t}}+\rho g\bar{\eta}+\bar{p}=0, \tag{4}$$

where D is the flexural rigidity of the ice sheet, $\bar{m}\left(=\rho_{\mathrm{i}}h\right)$ is mass per unit surface area of the ice sheet (ρ_{i} is the density of sea ice), $\bar{p}$ is the pressure acting at the lower surface of the ice sheet, and ρ is density of water. Here $\nabla^4_{\bar{x},\bar{y}}$ is the biharmonic operator in the plane of the ice sheet. The flexural rigidity D is usually related to the effective Young's modulus E and Poisson's ratio ν by the relation $D=Eh^3/12\left(1-\nu^2\right)$. The assumption of no cavitation between the ice and water at any time, gives the kinematic condition that

$$\left.\bar{\phi}_{\bar{z}}\right|_{\bar{z}=0}=\bar{\eta}_{\bar{t}}\quad\text{for}\quad \bar{y}>0 \tag{5}$$

while at the free surface region the velocity potential satisfies

$$\left.\bar{\phi}_{\bar{t}\bar{t}}\right|_{\bar{z}=0}+g\left.\bar{\phi}_{\bar{z}}\right|_{\bar{z}=0}=0\quad\text{for}\quad \bar{y}<0. \tag{6}$$

The two natural boundary conditions [15] at the edge of the ice sheet correspond to there being no bending and no shear there and are, respectively,

$$\left.\begin{array}{r}D\bar{\eta}_{\bar{y}\bar{y}}+D\nu\bar{\eta}_{\bar{x}\bar{x}}=0\\ D\nabla^2_{\bar{x},\bar{y}}\bar{\eta}_{\bar{y}}+D\left(1-\nu\right)\bar{\eta}_{\bar{y}\bar{x}\bar{x}}=0\end{array}\right\}\quad\text{for}\quad \bar{y}=0+,\bar{z}=0. \tag{7}$$

Eqns. 1 to 7, along with boundary conditions corresponding to there being an incident ocean wave of unit amplitude propagating from the open sea with angle θ to the y-axis, form the system that we will solve.

As mentioned previously a formal analytical solution of this boundary value problem was derived by Evans & Davies [4]. In their article, computation of the solution was not carried out because the solutions were considered to be unsuitable for numerical computation. In this paper we give a simplified version of the derivation of the solution by Evans & Davies, and give a few modifications that are needed to allow simple numerical computation of the solution.

3. Non-Dimensional Formulation

It is convenient to write the system in non-dimensional form by scaling distance by the characteristic length $l=\sqrt[4]{D/\rho g}$, and scaling time by the characteristic time $\sqrt{l/g}$. This scaling is optimal in the sense that it captures significant physical properties of the response of the system and hence the number of solutions required is reduced to a minimal, canonical, set [5]. The dimensionless quantities (without the overbar) are

$$\begin{array}{ccc}\left(x,y,z\right)=\left(\bar{x},\bar{y},\bar{z}\right)/l, & t=\bar{t}\sqrt{g/l}, & \omega=\bar{\omega}\sqrt{l/g},\\ \eta=\bar{\eta}/l, & \phi=\bar{\phi}/\left(l\sqrt{gl}\right), & \\ p=\bar{p}/\left(\rho gl\right), & m=\bar{m}/\left(\rho l\right). & \end{array} \tag{8}$$

Then, the equations for the nondimensional velocity potential become, from Eqns. from 1 to 6,

$$\begin{array}{lll} \left(\nabla^4_{x,y} + m\dfrac{\partial^2}{\partial t^2} + 1\right)\phi_z + \phi_{tt} = 0 & \text{for} & z = 0, y > 0, \\ \nabla^2_{x,y,z}\phi = 0 & \text{for} & -\infty < x, y < \infty, -H < z < 0, \\ \phi_z = 0 & \text{for} & z = -H, \\ \phi_{tt} = -\phi_z & \text{for} & z = 0, y < 0. \end{array} \tag{9}$$

The conditions at the ice edge, Eqns. 7, have the nondimensional form

$$\left.\begin{array}{r} \phi_{zyy} + \nu\phi_{zxx} = 0 \\ \nabla^2_{x,y}\phi_{zy} + (1-\nu)\,\phi_{zyxx} = 0 \end{array}\right\} \text{ for } y = 0+, z = 0. \tag{10}$$

4. Representation of the Solution

The incoming plane wave, which provides the forcing of the system, has x- and t- dependence, represented by (the real part of) $\exp ikx$ and $\exp i\omega t$, respectively, for some k and ω. Thus, the resulting potential can be expressed as

$$\phi(x, y, z, t) = \operatorname{Re}\left[\phi(y, z)\, e^{i(kx+\omega t)}\right]$$

where $\phi(y, z)$ is the complex potential satisfying the Helmholtz equation

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - k^2\right)\phi = 0 \quad \text{for} \quad -\infty < y < \infty, -H < z < 0 \tag{11}$$

with the boundary condition at the bottom of the sea

$$\left.\phi_z\right|_{z=-H} = 0. \tag{12}$$

Note that we use the same symbol for the real potential and the complex potential with the interpretation being taken from context.

Since the Helmholtz operator is separable in the y and z directions, and the eigenfunctions in the z-direction may be obtained from the z-dependence of the modes of the original self-adjoint system via a (bounded) projection operator, it follows that the eigenfunctions of the separated operator in the z-direction form a Riesz basis. (In fact, in the open-sea region these eigenfunctions are orthogonal.) Hence the solution $\phi(y, z)$ can be expressed in the separation-of-variables form as a sum of the functions $\exp(\pm i\alpha y)\exp(\pm\gamma z)$. Combining the $\exp(+\gamma z)$ and $\exp(-\gamma z)$ terms and applying the bottom condition 12 restricts the solution to a sum of modes with the form

$$\exp(\pm i\alpha y)\cosh\gamma(z+H). \tag{13}$$

The complex parameters (k, α, γ) must satisfy consistency conditions obtained by substituting Eqn. 13 into the remaining equations for $\phi(y, z)$,

$$\omega^2\phi - \phi_z = 0 \quad \text{for} \quad y < 0, z = 0, \tag{14}$$

$$\left(\left(\frac{\partial}{\partial y^2} - k^2\right)^2 - m\omega^2 + 1\right)\phi_z - \omega^2\phi = 0 \quad \text{for} \quad y > 0, z = 0, \tag{15}$$

giving

$$\gamma^2 = \alpha^2 + k^2, \tag{16}$$

$$f_{\text{sea}}(\gamma) = \omega^2\cosh\gamma H - \gamma\sinh\gamma H = 0, \tag{17}$$

$$f_{\text{ice}}(\gamma) = \omega^2\cosh\gamma H - \left(\gamma^4 + 1 - m\omega^2\right)\gamma\sinh\gamma H = 0. \tag{18}$$

Eqns. 17 and 18 are the *dispersion equations* for the open sea and ice-covered regions, respectively, and determine the relationship between the radial frequency ω and the wave number γ for a plane wave in the respective media.

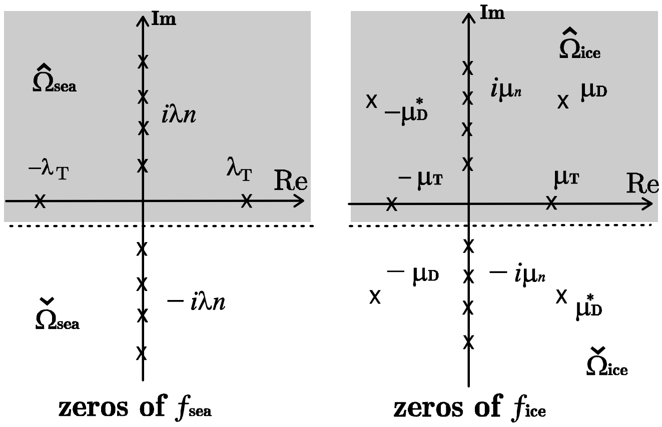

FIGURE 2. Schematic positions of the zeros of the dispersion equations f_{sea} and f_{ice}.

In appendix A we show that the open-sea dispersion equation 17 has two real roots that we denote $\pm\lambda_{\text{T}}$ ($\lambda_{\text{T}} > 0$) and a countably infinite set of pure imaginary roots denoted $\pm i\lambda_n$, $n = 1, 2, ...,$ ($\lambda_n > 0$). We also show that the ice-covered dispersion equation 18 has two real roots denoted $\pm\mu_{\text{T}}$ ($\mu_{\text{T}} > 0$), four complex roots denoted $\pm\mu_{\text{D}}$, $\pm\mu_{\text{D}}^*$ (μ_{D} has positive real and imaginary parts), and a countably infinite set of pure imaginary roots denoted $\pm i\mu_n$, $n = 1, 2, ...$ ($\mu_n > 0$). Figure 2 shows the a schematic of the positions of the roots, and the sets of roots $\hat{\Omega}_{\text{sea}}$ and $\check{\Omega}_{\text{sea}}$ which are the roots of f_{sea} with non-negative and negative imaginary part, respectively, while $\hat{\Omega}_{\text{ice}}$ and $\check{\Omega}_{\text{ice}}$ are the roots of f_{ice} with non-negative and negative imaginary part, respectively.

Causal solutions that satisfy radiation conditions can be expressed as the summation over modes of the system corresponding to roots with positive imaginary part or that are positive real (see [7]), as

$$\phi(y,z) = \left(Ie^{i\lambda_{\text{T}}' y} + Re^{-i\lambda_{\text{T}}' y}\right)\cosh\lambda_{\text{T}}(z+H) + \sum_{n=1}^{\infty} a_n e^{\lambda_n' y}\cos\lambda_n(z+H), \tag{19}$$

for $y < 0$ and

$$\begin{aligned}\phi(y,z) = {} & Te^{i\mu_{\text{T}}' y}\cosh\mu_{\text{T}}(z+H) + be^{i\mu_{\text{D}}' y}\cosh\mu_{\text{D}}(z+H) \\ & + b' e^{-i\mu_{\text{D}}'^* y}\cosh\mu_{\text{D}}^*(z+H) + \sum_{n=1}^{\infty} b_n e^{-\mu_n' y}\cos\mu_n(z+H),\end{aligned} \tag{20}$$

for $y > 0$. Here $T, I, R, a_n, b_n, b,$ and b' are complex coefficients of various modes. The coefficients $\lambda_{\text{T}}', i\lambda_n', \mu_{\text{T}}', \mu_{\text{D}}'$, and $i\mu_n'$ are wave numbers projected onto the y axis and are related to the roots of the dispersion equations by

$$\begin{aligned} \lambda_{\text{T}}' &= \sqrt{\lambda_{\text{T}}^2 - k^2}, \quad \lambda_n' = \sqrt{\lambda_n^2 + k^2}, \\ \mu_{\text{T}}' &= \sqrt{\mu_{\text{T}}^2 - k^2}, \quad \mu_{\text{D}}' = \sqrt{\mu_{\text{D}}^2 - k^2}, \quad \mu_n' = \sqrt{\mu_n^2 + k^2}, \end{aligned} \tag{21}$$

where the branch of the square root has been chosen so that the primed variables equal the unprimed roots when $k = 0$. Note that $\mu_{\text{D}}'^*$, $-\mu_{\text{D}}'$ and $-\mu_{\text{D}}'^*$ correspond to $-\mu_{\text{D}}^*$, $-\mu_{\text{D}}$ and $-\mu_{\text{D}}^*$, respectively.

Note that the modes with complex wave numbers μ_{D}' and $\mu_{\text{D}}'^*$ are exponentially decaying, with the decay faster than e^{-ky}. This follows since $\operatorname{Re}\mu_{\text{D}} < \operatorname{Im}\mu_{\text{D}}$ (cf.

appendix **??**) hence $\operatorname{Re}\mu_{\mathrm{D}}^2 < 0$ and $\operatorname{Re}\left(k^2 - \mu_{\mathrm{D}}^2\right) > k^2$. That is

$$\exp\left(iy\sqrt{\mu_{\mathrm{D}}^2 - k^2}\right) = \exp\left(-y\sqrt{k^2 - \mu_{\mathrm{D}}^2}\right) < A\exp\left(-ky\right).$$

Hence, only when $\mu_{\mathrm{T}} > k$, and correspondingly $\sqrt{\mu_{\mathrm{T}}^2 - k^2}$ is real, is there a wave that propagates through the ice sheet. When $\mu_{\mathrm{T}} < k$ all the wave modes in $y > 0$ are exponentially decaying, and hence there is no propagation of energy. Since $k = \lambda_{\mathrm{T}} \sin\theta$, where θ is the incidence angle, incidence at angles greater than the critical angle θ_{T}, for which $\mu_{\mathrm{T}} = \lambda_{\mathrm{T}} \sin\theta_{\mathrm{T}}$, cause total reflection of the wave energy as there can be no propagating wave in the ice sheet.

5. Derivation of the Wiener-Hopf Equation

The derivation of the solution given by Evans and Davies uses a Wiener-Hopf factorization and an application of Liouville's theorem which relies on the property that solutions 19 and 20 are,

$$\phi\left(y,z\right) = \begin{cases} O\left(1\right) & \text{as} \quad y \to -\infty \\ Te^{i\mu_{\mathrm{T}}' y}\cosh\mu_{\mathrm{T}}\left(z+H\right) + O\left(e^{-ky}\right) & \text{as} \quad y \to \infty \end{cases}$$

which follows from our observation that all non-travelling modes decay faster than e^{-ky}. Hence the function $\psi\left(y,z\right)$ defined as

$$\psi\left(y,z\right) = \phi\left(y,z\right) - Te^{i\mu_{\mathrm{T}}' y}\cosh\mu_{\mathrm{T}}\left(z+H\right),$$

i.e. the potential with the transmitted wave subtracted, behaves asymptotically as

$$\psi\left(y,z\right) = \begin{cases} O\left(1\right) & \text{as} \quad y \to -\infty \\ O\left(e^{-ky}\right) & \text{as} \quad y \to \infty \end{cases}. \tag{22}$$

It follows that the Fourier transform of $\psi\left(y,z\right)$ with respect to y,

$$\Psi\left(\alpha,z\right) = \int_{-\infty}^{\infty} \psi\left(y,z\right) e^{i\alpha y}\,\mathrm{d}\,y,$$

converges in $\mathcal{D} = \left\{\alpha \in \mathbb{C} : -k < \operatorname{Im}\alpha < 0\right\}$ and is a regular function of $\alpha \in \mathcal{D}$. Hence, $\psi\left(y,z\right)$ can be obtained by the inverse transform

$$\psi\left(y,z\right) = \frac{1}{2\pi}\int_{i\tau-\infty}^{i\tau+\infty} \Psi\left(\alpha,z\right) e^{-i\alpha y}\,\mathrm{d}\,y \tag{23}$$

for any τ, $-k < \tau < 0$ (cf. [14] chapter 2). Notice that the same argument and properties also apply to ψ_z and its Fourier transform.

The solution is obtained by the way of Fourier transforming the boundary conditions at the surface, open water and ice-covered regions, then deriving an algebraic expression for Ψ in the α-plane. Transforming Eqn. 11 and the bottom condition 12 give

$$\left(\frac{\partial^2}{\partial z^2} - \left(k^2 + \alpha^2\right)\right)\Psi\left(\alpha,z\right) = 0, \ \frac{\partial}{\partial z}\Psi\left(\alpha,-H\right) = 0. \tag{24}$$

We extend Eqn. 16 by setting $\gamma = \sqrt{\alpha^2 + k^2}$ with branch cuts in the α-plane stretching from $\pm ik$ to $\pm i\infty$. Then, $\operatorname{Re}\gamma > 0$ when $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\gamma < 0$ when $\operatorname{Re}\alpha < 0$. Thus, any solution of the initial value problem 24 can be expressed as

$$\Psi\left(\alpha,z\right) = \Psi\left(\alpha,0\right)\frac{\cosh\gamma\left(z+H\right)}{\cosh\gamma H}, \ \alpha \in \mathcal{D} \tag{25}$$

and also determines the relationship at the surface $z = 0$ that

$$\Psi_z\left(\alpha,0\right) = \Psi\left(\alpha,0\right)\gamma\tanh\gamma H, \ \alpha \in \mathcal{D}. \tag{26}$$

Later we will use the mapping from $\Psi_z\left(\alpha,0\right)$ to $\Psi\left(\alpha,z\right)$ defined by Eqns. 25 and 26.

For notational brevity we will write $\Psi'(\alpha)$ for $\Psi_z(\alpha,0)$. We define the one-sided Fourier transforms $\Psi_+(\alpha)$, $\Psi_-(\alpha)$, $\Psi'_+(\alpha)$, and $\Psi'_-(\alpha)$

$$\Psi_+(\alpha) = \int_0^\infty \psi(y,0)\, e^{i\alpha y}\, \mathrm{d}\, y, \Psi_-(\alpha) = \int_{-\infty}^0 \psi(y,0)\, e^{i\alpha y}\, \mathrm{d}\, y,$$
$$\Psi'_+(\alpha) = \int_0^\infty \psi_z(y,0)\, e^{i\alpha y}\, \mathrm{d}\, y, \Psi'_-(\alpha) = \int_{-\infty}^0 \psi_z(y,0)\, e^{i\alpha y}\, \mathrm{d}\, y.$$

It follows from Eqn. 22 that Ψ_+ and Ψ'_+ are regular in $\mathcal{D}_+ = \{\alpha \in \mathbf{C} : -k < \operatorname{Im}\alpha\}$, while Ψ_- and Ψ'_- are regular in $\mathcal{D}_- = \{\alpha \in \mathbf{C} : \operatorname{Im}\alpha < 0\}$. Note that $\Psi(\alpha,0) = \Psi_+(\alpha) + \Psi_-(\alpha)$ and $\Psi'(\alpha) = \Psi_z(\alpha,0) = \Psi'_+(\alpha) + \Psi'_-(\alpha)$. We will solve for $\Psi'(\alpha)$.

The one-sided transform of the surface equation 14 for ϕ gives the relation

$$\omega^2\Psi_-(\alpha) = \Psi'_- - \frac{iAT}{\mu'_{\mathrm{T}} + \alpha},\ \alpha \in \mathcal{D}_- \tag{27}$$

where

$$A = -\left(\mu_{\mathrm{T}}^4 - m\omega^2\right)\mu_{\mathrm{T}} \sinh \mu_{\mathrm{T}} H.$$

We have assumed that ψ_z is integrable and that ψ is bounded for $y \le 0$; we will see that both these assumptions hold later. Similarly, the transform of ice-covered surface equation 15 gives

$$\begin{aligned} \omega^2\Psi_+ &= \left(\gamma^4 + 1 - m\omega^2\right)\Psi'_+ - \left(c_3 - ic_2\alpha - \left(\alpha^2 + 2k^2\right)(c_1 - ic_0\alpha)\right) \\ &= \left(\gamma^4 + 1 - m\omega^2\right)\Psi'_+ - M_3,\ \alpha \in \mathcal{D}_+ \end{aligned} \tag{28}$$

where the four constants, c_i, $i = 1,2,3,4$, are

$$c_i = \left(\frac{\partial}{\partial y}\right)^i \psi_z \Bigg|_{y=0+,z=0}.$$

It is clear from the expansion 20 that each of ψ_{zy}, ψ_{zyy}, and ψ_{zyyy} is $O(\exp -ky)$ as $y \to \infty$. We will assume that ψ_{zyyyy} is integrable for $y \ge 0$, and justify this assumption later.

The sum of equations 27 and 28 gives a typical Wiener-Hopf equation

$$f_{\mathrm{ice}}(\gamma)\,\Psi'_+(\alpha) + f_{\mathrm{sea}}(\gamma)\,\Psi'_-(\alpha) + C(\alpha) = 0,\ \alpha \in \mathcal{D} \tag{29}$$

where

$$C(\alpha) = \left(\frac{iAT}{\mu'_{\mathrm{T}} + \alpha} + M_3(\alpha)\right)\gamma \sinh \gamma H.$$

We have used Eqn. 26 to replace Ψ by expressions in Ψ' to arrive at Eqn. 29. Note that since the functions Ψ'_+ and Ψ'_- are regular in $\mathcal{D}_+$ and $\mathcal{D}_-$, respectively, all functions in Eqn. 29 are regular and non-zero in the strip $\mathcal{D} = \mathcal{D}_+ \cap \mathcal{D}_-$.

6. Solution of the Wiener-Hopf Equation

Equation 29 may be solved for $\Psi'(\alpha)$ by equating the ratio $f_{\mathrm{ice}}/f_{\mathrm{sea}}$ to the ratio K_+/K_-, where K_+ and K_- are regular non-zero functions in $\mathcal{D}_+$ and $\mathcal{D}_-$, respectively. The decomposition is easily achieved by expressing $f_{\mathrm{ice}}/f_{\mathrm{sea}}$ as an infinite products of simple polynomials with roots given in Eqn. 21 by applying

Weierstrass's factor theorem ([2] section 2.9). Hence K_+ and K_- are

$$K_+(\alpha) = \prod_{q\in\check{\Omega}_{\mathrm{sea}}} \frac{q}{q'-\alpha} \prod_{q\in\check{\Omega}_{\mathrm{ice}}} \frac{q'-\alpha}{q},$$

$$K_-(\alpha) = \prod_{q\in\hat{\Omega}_{\mathrm{sea}}} \frac{q'-\alpha}{q} \prod_{q\in\hat{\Omega}_{\mathrm{ice}}} \frac{q}{q'-\alpha},$$

where $q' = \sqrt{q^2-k^2}$ as in Eqn. 21, and $\check{\Omega}_{\mathrm{sea}}$, $\hat{\Omega}_{\mathrm{sea}}$, $\check{\Omega}_{\mathrm{ice}}$, $\hat{\Omega}_{\mathrm{ice}}$ are the sets of roots defined previously. It is clear that K_+ and K_- are indeed regular non-zero in $\mathcal{D}_+$ and $\mathcal{D}_-$, respectively.

Notice that

$$\lambda_n = \pi n/H + O\left(n^{-1}\right),\ \mu_n = \pi n/H + O\left(n^{-5}\right)$$

as $n\to\infty$, and that the part of the infinite products of K_+ over the pure imaginary roots can alternatively be expressed as

$$\prod_{n=1}^{\infty} \frac{\sqrt{1+\frac{k^2}{\mu_n^2}} - \frac{i\alpha}{\mu_n}}{\sqrt{1+\frac{k^2}{\lambda_n^2}} - \frac{i\alpha}{\lambda_n}} = \prod_{n=1}^{\infty}\left(\frac{\lambda_n}{\mu_n}\right)\prod_{n=1}^{\infty}\frac{\sqrt{\mu_n^2+k^2}-i\alpha}{\sqrt{\lambda_n^2+k^2}-i\alpha}$$

Since $\lambda_n/\mu_n = 1+O\left(n^{-2}\right)$ we see that $\prod_{n=1}^{\infty}\lambda_n/\mu_n$ converges. Also

$$\prod_{n=1}^{\infty}\frac{\sqrt{\mu_n^2+k^2}-i\alpha}{\sqrt{\lambda_n^2+k^2}-i\alpha} = \prod_{n=1}^{\infty}(1+g_n), \tag{30}$$

where

$$g_n(\alpha) = \frac{\sqrt{\mu_n^2+k^2}-\sqrt{\lambda_n^2+k^2}}{\sqrt{\lambda_n^2+k^2}-i\alpha}.$$

Since $g_n(\alpha)\to 0$ as $|\alpha|\to\infty$, $\alpha\in\mathcal{D}_+$ and $|g_n(\alpha)| = O\left(n^{-2}\right)$, the infinite products 30 tend to 1 as $|\alpha|\to\infty$ in $\mathcal{D}_+$. Similarly, the infinite products in K_- are also $O(1)$. Hence

$$\begin{aligned} K_+(\alpha) &= O\left(\alpha^2\right) \quad \text{as} \quad |\alpha|\to\infty, \alpha\in\mathcal{D}_+,\\ K_-(\alpha) &= O\left(\alpha^{-2}\right) \quad \text{as} \quad |\alpha|\to\infty, \alpha\in\mathcal{D}_-. \end{aligned}$$

Using the relationship

$$f_{\mathrm{ice}} - f_{\mathrm{sea}} = -\left(\gamma^4 - m\omega^2\right)\gamma\sinh\gamma H,$$

the Wiener-Hopf equation 29 becomes

$$\begin{aligned} &f_{\mathrm{ice}}\left\{\left(\gamma^4-m\omega^2\right)\Psi'_+ - \tfrac{iAT}{\mu'_{\mathrm{T}}+\alpha} - M_3(\alpha)\right\}\\ &= -f_{\mathrm{sea}}\left\{\left(\gamma^4-m\omega^2\right)\Psi'_- + \tfrac{iAT}{\mu'_{\mathrm{T}}+\alpha} + M_3(\alpha)\right\}. \end{aligned}$$

Thus, substituting K_+/K_- for $f_{\mathrm{ice}}/f_{\mathrm{sea}}$ gives the Wiener-Hopf factorization

$$\begin{aligned} &K_+\left\{\left(\gamma^4-m\omega^2\right)\Psi'_+ - M_3(\alpha)\right\} - \tfrac{iAT\left(K_+(\alpha)-K_+\left(-\mu'_{\mathrm{T}}\right)\right)}{\mu'_{\mathrm{T}}+\alpha}\\ &= -K_-\left\{\left(\gamma^4-m\omega^2\right)\Psi'_- + M_3(\alpha)\right\} - \tfrac{iAT\left(K_-(\alpha)-K_+\left(-\mu'_{\mathrm{T}}\right)\right)}{\mu'_{\mathrm{T}}+\alpha}. \end{aligned} \tag{31}$$

Note that $\dfrac{-iATK_+\left(-\mu'_{\mathrm{T}}\right)}{\mu'_{\mathrm{T}}+\alpha}$ has been added to the both sides in order to avoid $\alpha = -\mu'_{\mathrm{T}}$ being a singularity in $\mathcal{D}_+$.

From equation 28, the left hand side of equation 31 is

$$K_+\left\{\omega^2\Psi_+ - \Psi'_+\right\} - \frac{iAT\left(K_+(\alpha) - K_+(-\mu'_{\mathrm{T}})\right)}{\mu'_{\mathrm{T}} + \alpha}.$$

Since ψ and ψ_z are bounded for $y \geq 0$, $\Psi_+, \Psi'_+ \to 0$ as $|\alpha| \to \infty$ in $\mathcal{D}_+$. It has been shown that $K_+(\alpha) = O\left(\alpha^2\right)$ as $|\alpha| \to \infty$ in $\mathcal{D}_+$, and $M_3 = O\left(\alpha^3\right)$. Thus the left hand side of equation 31 is $o\left(\alpha^2\right)$ as $|\alpha| \to \infty$ in $\mathcal{D}_+$. Similarly, the right hand side of equation 31 is also $o\left(\alpha^2\right)$ (cf. [14]).

Each of the right and the left hand sides of equation 31 are analytic in $\mathcal{D}$, thus through analytic continuation, equation 31 defines a function $J(\alpha)$ regular in the whole plane. Furthermore, since each side of Eqn. 31 is $o\left(\alpha^2\right)$ as $|\alpha| \to \infty$, Liouville's theorem guarantees that $J(\alpha)$ is a polynomial of degree one, i.e. $J(\alpha) = a_1\alpha + a_0$. Equating each side of equation 31 to J allows us to solve for Ψ'_+ and Ψ'_- from which it follows that $\Psi'(\alpha)$ is given by

$$\left(\gamma^4 - m\omega^2\right)\Psi'(\alpha) = \left(J(\alpha) - \frac{iATK_+(-\mu'_{\mathrm{T}})}{\mu'_{\mathrm{T}} + \alpha}\right)\left(\frac{1}{K_+(\alpha)} - \frac{1}{K_-(\alpha)}\right).$$

Using Eqns. 25 and 26 we extend Ψ' to give the alternative expressions for $\Psi(\alpha, z)$

$$\Psi(\alpha, z) = \frac{F(\alpha)\cosh\gamma(z+H)}{K_+(\alpha)\, d_{\mathrm{sea}}(\gamma)\cosh\gamma H} = \frac{F(\alpha)\cosh\gamma(z+H)}{K_-(\alpha)\, d_{\mathrm{ice}}(\gamma)\cosh\gamma H} \tag{32}$$

where

$$F(\alpha) = a_1\alpha + a_0 - \frac{iATK_+(-\mu'_{\mathrm{T}})}{\mu'_{\mathrm{T}} + \alpha},$$
$$d_{\mathrm{sea}}(\gamma) = \omega^2 - \gamma\tanh\gamma H,$$
$$d_{\mathrm{ice}}(\gamma) = \omega^2 - \left(\gamma^4 + 1 - m\omega^2\right)\gamma\tanh\gamma H.$$

Since Ψ is $O\left(\alpha^{-2}\right)$ as $|\alpha| \to \infty$, the inverse transform can be calculated by closing the contour of integration in either $\mathcal{D}_+$ or $\mathcal{D}_-$.

7. Solution of the BVP

Since $\Psi(\alpha, z)$ is singular in α only at the roots of d_{sea} and d_{ice} and $\alpha = -\mu'_{\mathrm{T}}$, $\Psi(\alpha, z)$ can be written as a Mittag-Leffler [2] expansion over either of the sets of roots $\Omega_{\mathrm{sea}} = \hat{\Omega}_{\mathrm{sea}} \cup \check{\Omega}_{\mathrm{sea}}$, or $\Omega_{\mathrm{ice}} = \hat{\Omega}_{\mathrm{ice}} \cup \check{\Omega}_{\mathrm{ice}} \cup \{-\mu'_{\mathrm{T}}\}$. Those expansions are

$$\Psi(\alpha, z) = \begin{cases} \dfrac{F(\alpha)}{K_+(\alpha)} \displaystyle\sum_{q\in\Omega_{\mathrm{sea}}} \dfrac{\cosh q(z+H)}{\cosh qH}\,\dfrac{qR_{\mathrm{sea}}(q)}{q'(\alpha - q')} \\ \dfrac{F(\alpha)}{K_-(\alpha)} \displaystyle\sum_{q\in\Omega_{\mathrm{ice}}} \dfrac{\cosh q(z+H)}{\cosh qH}\,\dfrac{qR_{\mathrm{ice}}(q)}{q'(\alpha - q')} \end{cases}$$

where the residues R_{sea} and R_{ice} of $1/d_{\mathrm{sea}}$ and $1/d_{\mathrm{ice}}$, respectively, are

$$R_{\mathrm{sea}}(q) = \left[-\tanh qH - qH\left(1 - \tanh^2 qH\right)\right]^{-1},$$
$$R_{\mathrm{ice}}(q) = \left[qH\left(q^4 + 1 - m\omega^2\right)\left(\tanh^2 qH - 1\right) - \left(5q^4 + 1 - m\omega^2\right)\tanh qH\right]^{-1}.$$

The integration in Eqn. 23 may then be obtained by the summation over the residues of the integrand in either $\mathcal{D}_-$ or $\mathcal{D}_+$.

For $y < 0$ we complete the contour in $\mathcal{D}_+$ giving

$$\psi(y, z) = -Te^{i\mu'_{\mathrm{T}} y}\cosh\mu_{\mathrm{T}}(z+H) + \sum_{q\in\hat{\Omega}_{\mathrm{sea}}} \frac{iF(q')\, qR_{\mathrm{sea}}(q)}{q'K_+(q')} e^{-iq'y}\frac{\cosh q(z+H)}{\cosh qH} \tag{33}$$

while for $y > 0$ we complete the contour in $\mathcal{D}_-$ giving

$$\psi(y,z) = -\sum_{q\in\check{\Omega}_{\text{ice}}} \frac{iF(q')\,qR_{\text{ice}}(q)}{q'K_-(q')} e^{-iq'y} \frac{\cosh q(z+H)}{\cosh qH}. \tag{34}$$

Note that the coefficients in these solutions are $O(q^{-2})$ as $|q| \to \infty$, thus the infinite summations in equations 33 and 34 converge for any y and z.

Solutions 33 and 34 contain the two unknown constants a_1 and a_0, which may be determined by substituting solution 34 into the edge conditions 10. Rewriting equations 10 for $\psi(y,z)$ gives

$$\begin{cases} \psi_{zyy} - \nu k^2 \psi_z = T\mu_{\text{T}} \left(\mu_{\text{T}}'^2 + \nu k^2\right) \sinh \mu_{\text{T}} H \\ \psi_{zyyy} - (2-\nu) k^2 \psi_{zy} = iT\mu_{\text{T}}\mu_{\text{T}}' \left(\mu_{\text{T}}'^2 + (2-\nu) k^2\right) \sinh \mu_{\text{T}} H \end{cases} \tag{35}$$

for $y = 0+$, $z = 0$,giving two equations for a_0 and a_1 after substituting solution 34. The derivatives of ψ can be expressed as

$$\left(\frac{\partial}{\partial y}\right)^n \psi_z(0+,0) = A_n \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} + b_n, \; n = 0,1,2,3,$$

where A_n is the row vector defined by

$$(A_n)_m = \sum_{q\in\check{\Omega}_{\text{ice}}} (-iq')^n q'^m G(q,q')$$

and b_n is the scalar

$$b_n = -iATK_+(-\mu_{\text{T}}') \sum_{q\in\check{\Omega}_{\text{ice}}} \frac{(-iq')^n G(q,q')}{q' + \mu_{\text{T}}'}$$

where

$$G(q,q') = \frac{-iq^2 R_{\text{ice}}(q)}{q'K_-(q')} \tanh qH.$$

Hence, from equations 35, coefficients a_0 and a_1 are given by

$$\begin{aligned} \begin{bmatrix} A_2 - \nu k^2 A_0 \\ A_3 - (2-\nu) k^2 A_1 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} &= T\mu_{\text{T}} \sinh \mu_{\text{T}} H \begin{pmatrix} \mu_{\text{T}}'^2 + \nu k^2 \\ i\mu_{\text{T}}' \left(\mu_{\text{T}}'^2 + (2-\nu) k^2\right) \end{pmatrix} \\ &\quad - \begin{pmatrix} b_2 - \nu k^2 b_0 \\ b_3 - (2-\nu) k^2 b_1 \end{pmatrix}. \end{aligned} \tag{36}$$

Given a_1 and a_0 from Eqn. 36, one can straightforwardly calculate the solution in the region $y < 0$ using Eqn. 33, or in the region $y > 0$ using Eqn. 34.

8. Polynomial Form of Coefficients

The formulas for solution in Eqns. 33 and 34 contain exponentially growing functions which lead to round-off problems and sensitivity to numerical errors in roots and as ω or H become large. Forms more suitable for numerical computation may be found by using Eqns. 17 and 18 to substitute polynomials for the exponential functions. The resulting rational polynomial forms not only allow robust numerical calculation, but also allow us to easily establish the boundedness and integrability conditions assumed for the existence of the Fourier transformed functions.

A further improvement in numerical stability may be achieved by setting the transmitted wave amplitude T equal to $1/\cosh \mu_{\text{T}} H$, thereby stabilizing computation of the two constants a_1 and a_0. Using Eqn. 18 to make the substitution $\tanh qH = \omega^2/q\left(q^4 - m\omega^2 + 1\right)$, for $q \in \Omega_{\text{ice}}$ then simplifies the term $T\mu_{\text{T}} \sinh \mu_{\text{T}} H$ in Eqn. 36 to $-\omega^2\left(\mu_{\text{T}}^4 - m\omega^2\right)/\left(\mu_{\text{T}}^4 - m\omega^2 + 1\right)$.

The same substitution in the formulas for R_{sea}, R_{ice}, and G give the computationally stable forms

$$R_{\text{sea}}(q) = \frac{-q}{\omega^2 + H\left(q^2 - \omega^4\right)},$$

$$R_{\text{ice}}(q) = \frac{-q\left(q^4 - m\omega^2 + 1\right)}{H\left\{q^2\left(q^4 - m\omega^2 + 1\right)^2 - \omega^4\right\} + \omega^2\left(5q^4 - m\omega^2 + 1\right)},$$

$$G(q, q') = \frac{-i\mu R_{\text{ice}}(q)}{q' K_-(q')} \frac{\omega^2}{\mu^4 - m\omega^2 + 1}.$$

Notice that G is $O\left(q'^{-7}\right)$ as $|q'| \to \infty$ so that the infinite summations in equation 36 converge rapidly.

Note that all coefficients of the various modes have been expressed as rational polynomials of the roots of the dispersion equations for the free surface and the ice-covered region. By examining the order of these coefficients in the solution, boundedness of derivatives of the solution can be established. For example, consider the expression for $\psi_z(y, 0)$, $y \leq 0$, in equation 33. The part of that expression that is the summation over pure imaginary roots $\{\lambda_n\}$ becomes

$$\sum_{n=1}^{\infty} \frac{iF\left(i\lambda'_n\right)\lambda_n R_{\text{sea}}(i\lambda_n)}{\lambda'_n K_+\left(i\lambda'_n\right)} e^{\lambda'_n y} \lambda_n \tan \lambda_n H,$$

and hence, using the polynomial substitution for $\tan \lambda_n H$ as above, the coefficients in this summation can be seen to be $O\left(\lambda_n^{-2}\right)$ for large n. Similarly in the expression for $\psi_z(y, 0)$, $y \geq 0$, the coefficients of the summation are found to be $O\left(\mu_n^{-6}\right)$. It follows that each derivative of $\psi_z(y, 0)$ with respect to y, up to the fourth derivative, is bounded for any $y \geq 0$. Hence, the assumptions made in taking the Fourier transform of equations 14 and 15 are justified.

It remains to establish the integrability of $\psi_z(y, 0)$ for $y \leq 0$ and $\psi_{zyyyy}(y, 0)$ for $y \geq 0$. Evans & Davies [4] studied the transform function $\Psi(\alpha, 0)$ for $|\alpha| \to$, and consequently claimed that those two functions have, at worst, a log-like singularity at $y = 0$. However, the solution expressed by the polynomials of the roots of the dispersion equations reveals that each of $\psi_z(y, 0)$, $y \leq 0$, and $\psi_{zyyyy}(y, 0)$, $y \geq 0$, are actually bounded.

In a physical sense, the biharmonic term of the plate equation for the vertical displacement is associated with the potential energy due to the bending of the plate (see [15] section 6.4). Hence boundedness of energy requires that all derivatives up to the fourth derivative of the displacement function should be bounded, as it has been confirmed.

9. Reflection and Transmission Coefficients

The reflection coefficient $\mathcal{R}$ and the transmission coefficient $\mathcal{T}$ for wave amplitude, given by

$$\mathcal{R} = \frac{|R|}{|I|} \quad \text{and} \quad \mathcal{T} = \frac{\mu_{\text{T}} \sinh \mu_{\text{T}} H}{\lambda_{\text{T}} \sinh \lambda_{\text{T}} H} \frac{|T|}{|I|},$$

can also be expressed without the exponential functions.

Recall that we have set $T = 1/\cosh \mu_{\text{T}} H$. The incident wave amplitude $|I|$ can be written as

$$|I| = \left| \frac{F\left(-\lambda'_{\text{T}}\right)\lambda_{\text{T}} R_-(\lambda_{\text{T}})}{-\lambda'_{\text{T}} K_+\left(-\lambda'_{\text{T}}\right)\cosh \lambda_{\text{T}} H} \right|$$

Thus, the transmission coefficient can be written simply as

$$\mathcal{T} = \frac{1}{|\mu_{\mathrm{T}}^4 - m\omega^2 + 1|} \left| \frac{-\lambda'_{\mathrm{T}} K_+ \left(-\lambda'_{\mathrm{T}}\right)}{F\left(-\lambda'_{\mathrm{T}}\right) \lambda_{\mathrm{T}} R_- \left(\lambda_{\mathrm{T}}\right)} \right|.$$

Using the identity $\left|K_+\left(\lambda'_{\mathrm{T}}\right)/K_+\left(-\lambda'_{\mathrm{T}}\right)\right| = 1$, the reflection coefficient can be simply computed by

$$\mathcal{R} = \left| \frac{F\left(\lambda'_{\mathrm{T}}\right)}{F\left(-\lambda'_{\mathrm{T}}\right)} \right|,$$

which again does not contain any exponentials.

Figure 3 shows computed values of the reflection coefficient when the non-dimensional water depth is $H = 2\pi$ for incidence angles from 0 to 90 degrees, and for a range of non-dimensional frequencies.In the numerical computation we

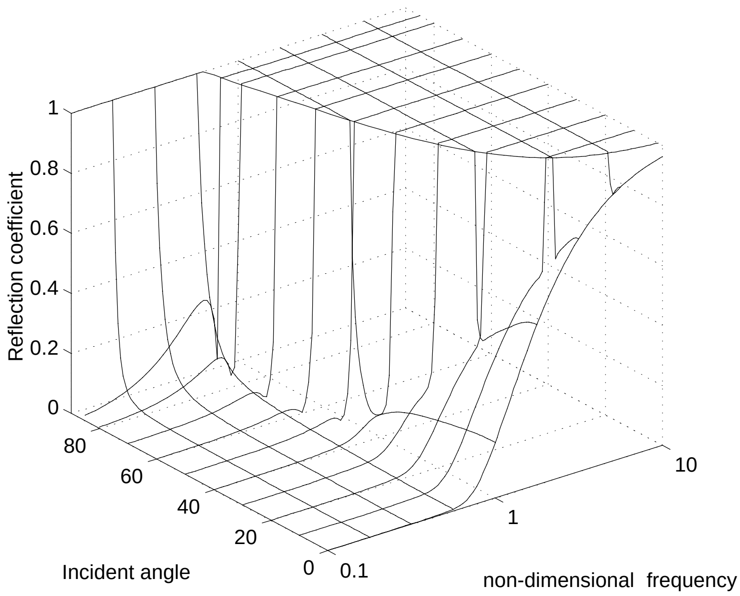

FIGURE 3. Three dimensional plot of the reflection coefficient for incident wave frequency and incident angle. The water depth is $H = 2\pi$.

have used the typical physical parameters $E = 6 \times 10^9\,\mathrm{Pa}$, $\nu = 0.3$, $g = 9.8\,\mathrm{m\,s^{-2}}$, $\rho = 1025.0\,\mathrm{kg\,m^{-3}}$, $\rho_{\mathrm{i}} = 922.5\,\mathrm{kg\,m^{-3}}$, $h = 1.0\,\mathrm{m}$.

10. Conclusions

We have shown that the solutions of the boundary problem modelling wave-ice interaction can be derived analytically using the Wiener-Hopf technique and those solutions may be easily computed. Our purpose in reintroducing a technique first published over 30 years ago is to show that, contrary to long held belief, computation of the coefficients of the solution is simple and efficient once the positions of the roots of the dispersion equations have been computed. An important feature of our solution method, that is perhaps not obvious, is that the ice-edge and radiation boundary conditions do not appear in the in the derivation of the general solutions in Eqns. 33 and 34. Consequently, solutions under different ice-edge and forcing conditions may be easily found using the solution method we have given, by imposing the desired boundary conditions as we do in section 7.

Another variation of the Wiener-Hopf technique, also building on the earlier work of Evans & Davies [4], has been presented by Balmforth & Craster [1]. Our view is that the derivation given by Balmforth & Craster was unnecessarily complicated compared to that of Evans & Davies who incorporated the natural boundary conditions as a system of two algebraic equations, as we do in this paper. The derivation we give here is further simplified by first establishing the expansion of the solution over modes given by the roots of dispersion equation, thereby greatly simplifying the establishment of asymptotic results required.

Our main contribution has been to show that by first finding the roots of the dispersion equations, application of the Wiener-Hopf technique becomes straightforward because the asymptotic results and factorizations follow from simple observations about the positions of the roots. Further, computation of the inverse transforms required to compute solutions, a traditional difficulty in applying the Wiener-Hopf method, was reduced to a straightforward sum over modes. We conclude that similar systems of partial differential equations, amenable to solution by the Wiener-Hopf technique, may also be solved in a simple manner once the roots of the appropriate dispersion equations are computed.

Appendix A. Locating the roots of the dispersion equations

As we have shown, knowledge of the location of the roots of the dispersion equations is the only problem-dependent step that needs to be performed. Once the location of the roots is known the Wiener-Hopf factorization and application of Liouville's theorem was straightforward, as is the calculation of the resulting inverse transform giving the solution. Because finding these roots is the key problem-dependent step in our solution we now show how the roots may be generally located to allow subsequent numerical evaluation.

The dispersion equation for the ice-covered sea, given in equation 18, is

$$f_{\text{ice}}(\gamma) = \omega^2 \cosh \gamma H - \left(\gamma^4 + 1 - m\omega^2\right) \gamma \sinh \gamma H = 0.$$

One may find real and pure imaginary roots easily by slight rearrangement of this equation.

Any real roots γ must satisfy

$$\tanh \gamma H = \frac{\omega^2}{\gamma^5 + \gamma\left(1 - m\omega^2\right)}.$$

Figure 4 shows a plot of the functions $\tanh \gamma H$ and $\omega^2 / \left(\gamma^5 + \gamma\left(1 - m\omega^2\right)\right)$ for the case $\left(1 - m\omega^2\right) > 0$. When $\left(1 - m\omega^2\right) < 0$ the polynomial term is negative for $0 < \gamma < \sqrt[4]{m\omega^2 - 1}$. In either case, the polynomial term and the hyperbolic tangent term intersect exactly once for $\gamma > 0$, and since each function is odd in γ there are always exactly two real roots occurring as plus and minus some positive value which we denote μ_{T}, where

$$\mu_{\mathrm{T}} > \begin{cases} 0 & \text{if} \quad \left(1 - m\omega^2\right) \geq 0 \\ \sqrt[4]{m\omega^2 - 1} & \text{if} \quad \left(1 - m\omega^2\right) < 0 \end{cases}.$$

The lower bound given may be used as the starting point for a numerical root-finding procedure.

When γ is pure imaginary, i.e. $\gamma = i\mu$ for some real μ, the root satisfies

$$\tan \mu H = \frac{-\omega^2}{\mu^5 + \mu\left(1 - m\omega^2\right)}.$$

Figure 5 shows a plot of the functions $\tan \mu H$ and $-\omega^2 / \left(\mu^5 + \mu\left(1 - m\omega^2\right)\right)$ again for the case $\left(1 - m\omega^2\right) > 0$. In this case, because the polynomial term is always negative, it is clear that each branch of the tan function, except the branch that

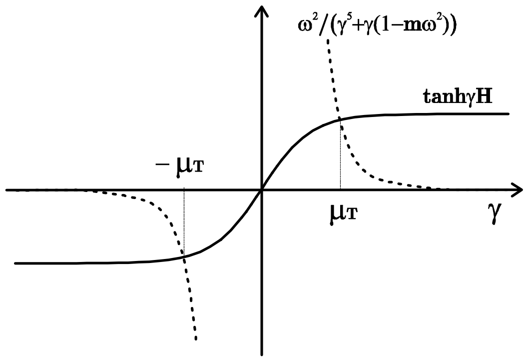

FIGURE 4. Plot of $\tanh \gamma H$ (solid) and $\omega^2 / \left(\gamma^5 + \gamma\left(1 - m\omega^2\right)\right)$ (dotted) when $1 > m\omega^2$, with no particular vertical and horizontal scale.

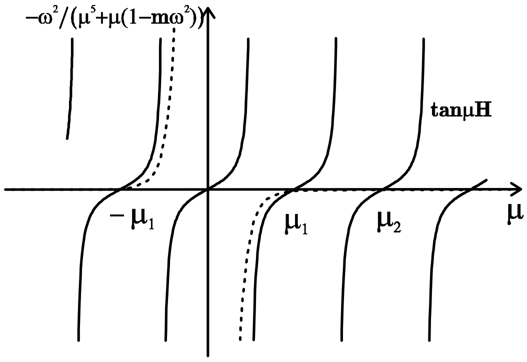

FIGURE 5. Plot of $\tan \mu H$ (solid) and -$\omega^2 / \left(\mu^5 + \mu\left(1 - m\omega^2\right)\right)$ (dotted) when $1 > m\omega^2$, with no particular vertical and horizontal scale.

passes through the origin, intersects the polynomial term exactly once and hence the n^{th} positive (imaginary) root satisfies $(n - 1/2)\,\pi < \mu_n H < n\pi$, $n = 1, 2, 3, \cdots$, with $\mu_n H \to n\pi$ as $n \to \infty$. Again, since the functions are odd, roots also occur at $-\mu_n$, $n = 1, 2, 3, \cdots$. The case $\left(1 - m\omega^2\right) > 0$ is typical for frequencies and ice thicknesses of geophysical importance, however the case $\left(1 - m\omega^2\right) < 0$ may occur at very high frequencies in which case the polynomial term is positive for $\gamma H < \sqrt[4]{m\omega^2 - 1}$ and we have the more general bound

$$\begin{array}{ccc} (n-1)\,\pi < \mu_n H < (n-1/2)\,\pi & \text{if} & \sqrt[4]{m\omega^2 - 1} \geq (n-1/2)\,\pi/H \\ (n-1/2)\,\pi < \mu_n H < n\pi & \text{if} & \sqrt[4]{m\omega^2 - 1} < (n-1/2)\,\pi/H \end{array} .$$

These bounds provide an initial bracket suitable for initializing a numerical procedure to evaluate each pure imaginary root.

We can check for any remaining roots using the argument principle [3]. Consider the change in argument of the rearranged dispersion equation

$$f_1(\gamma) = \mathrm{e}^{2\gamma H} - \frac{\gamma^5 + \gamma\left(1 - m\omega^2\right) + \omega^2}{\gamma^5 + \gamma\left(1 - m\omega^2\right) - \omega^2}$$

as γ is taken anti-clockwise on the square with vertices $(\pm(N + 1/4)\,\pi/H, \pm(N + 1/4)\,\pi/H)$. For large enough integer N it is easy to see that the change in argument is $4N\pi + 2\pi$ and so the number of zeros of f_1 minus the number of zeros of f_1 enclosed in the square is $2N+1$. Since zeros of f_1 are also zeros of f_{ice}, and since f_1 has 5 poles, the number of zeros of the dispersion equation that lie within the square is $2N+6$. There are $2N+2$ real or pure imaginary zeros, found above, and hence there must be 4 extra complex roots. Since f_{ice} is even and has real coefficients it follows that if μ_{D} is a root then so are $-\mu_{\text{D}}$, μ_{D}^*, and $-\mu_{\text{D}}^*$. Note that μ_{D} can not be zero except possibly when $\omega = 0$. So we may take μ_{D} to have positive real and imaginary parts. Consider now the change in argument of the function

$$f_2(\gamma) = \tan(\mu H)\left(\mu^5 + \mu\left(1 - m\omega^2\right)\right) + \omega^2$$

on the triangular contour with vertices $(0,0)$ and $((N+1/4)\pi/H, \pm(N+1/4)\pi/H)$. The change of argument establishes that the number of zeros of f_2 minus the number of zeros of f_2 enclosed in the triangle is 2. We found N real zeros, above, and since the tan function has N real poles, there are two zeros other than the real ones. Since zeros of f_2 are $-\text{i}$ times the roots of f_{ice}, we find that $\text{Im}(\mu_{\text{D}}) > \text{Re}(\mu_{\text{D}})$. This root may be computed using a fixed-point iteration scheme.

The dispersion equation for the open sea may also be analyzed in this way with the primary difference being that the fourth-order term in γ that appears in f_{ice} does not appear in f_{sea}. Thus, the open sea dispersion equation has the same structure of real and pure imaginary roots as f_{ice}, but does not have the four extra complex roots.

Also note that our arguments, above, have also established that each root of the dispersion equations is simple, i.e. has multiplicity one.

A listing of computer codes (in MatLab) to compute these roots can be found in Chung and Fox 1998 [6], appendix C.

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