The University of Auckland – Applied Mathematics
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Statistical Solutions to Inverse Problems : some examples

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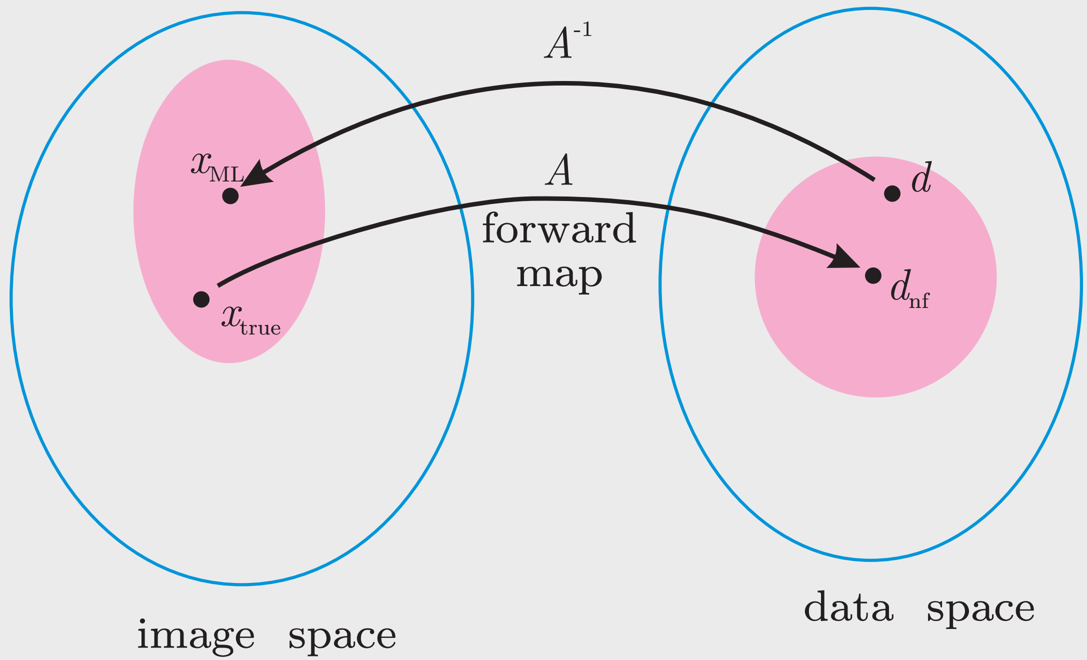
In this talk

- Inferential formulation
- What problem are we trying to solve? (Questions and answers)
- Some inverse problems and image models
- A taste of the details

Bayesian Formulation for Inverse problems

$d = Ax + n$: data d, image x, measurement noise n, forward map A



Posterior distribution for x conditional on d

$$\pi(x|d, m) \propto \Pr(d|x, m) \Pr(x|m) \qquad \text{(Bayes' rule)}$$

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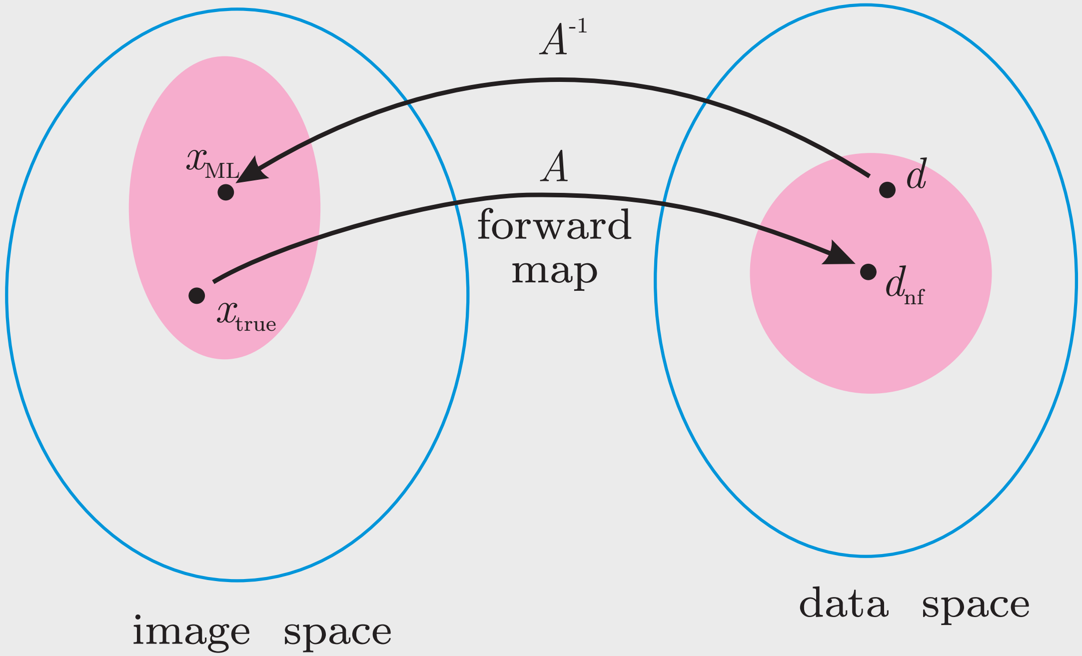

Posterior distribution for x conditional on d

$$\pi(x|d,m) \propto \color{red}{\Pr(d|x,m)}\Pr(x|m) \qquad \text{(Bayes' rule)}$$

Likelihood determined by measurement and noise process

Bayesian Formulation for Inverse problems

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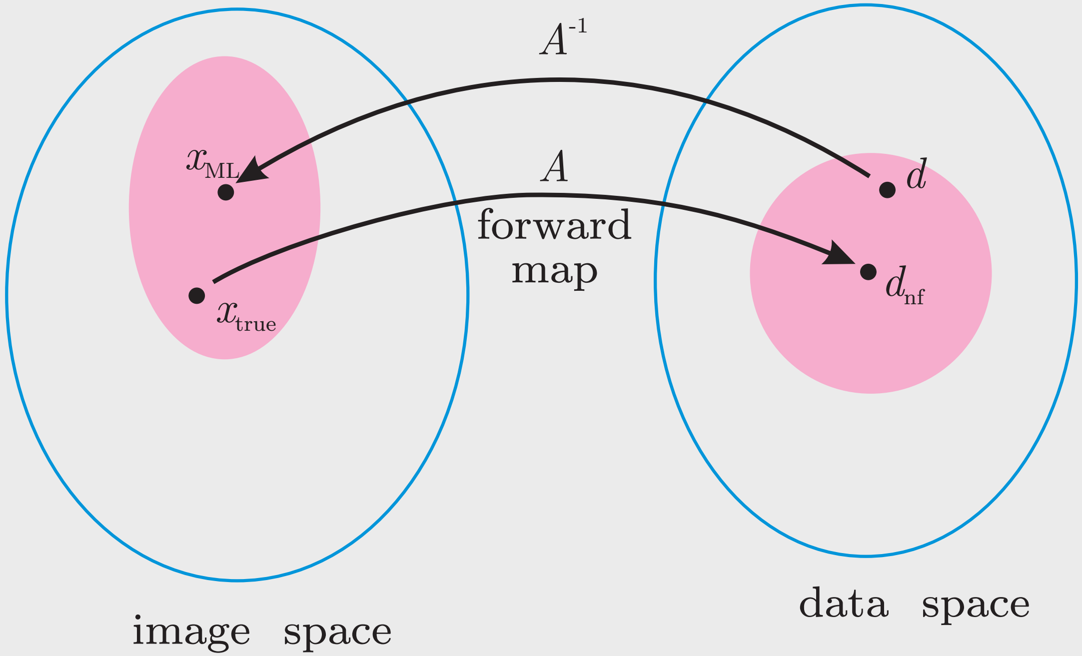

Posterior distribution for x conditional on d

$$\pi\left(x|d,m\right) \propto \Pr\left(d|x,m\right)\Pr\left(x|m\right) \qquad \text{(Bayes' rule)}$$

Likelihood determined by measurement and noise process

Prior, state space determined by modelling

Bayesian Formulation for Inverse problems

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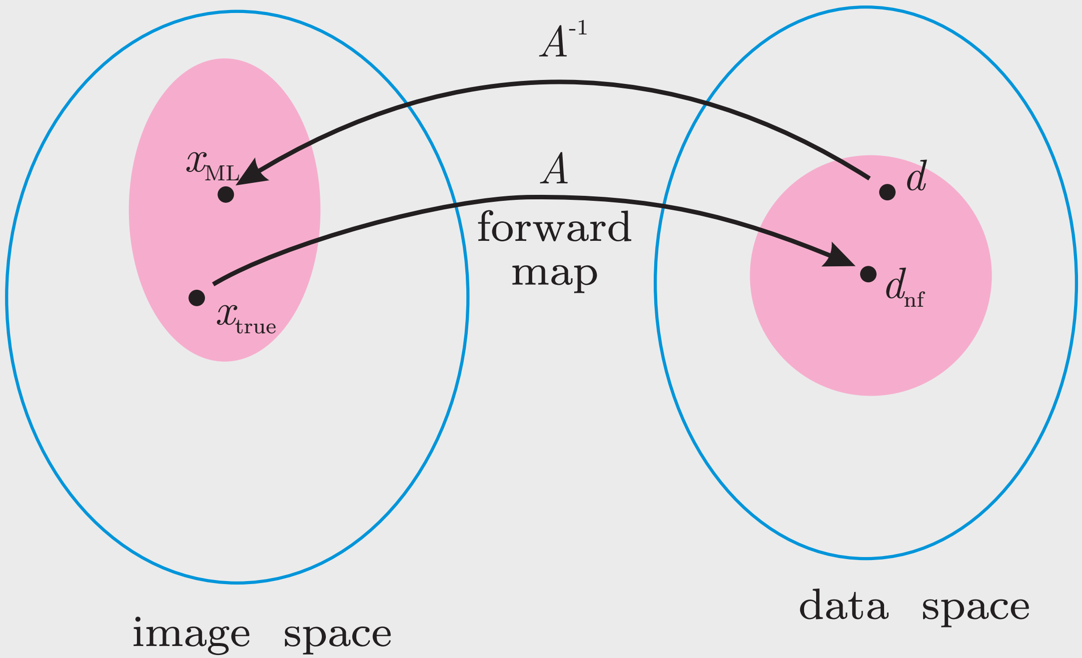


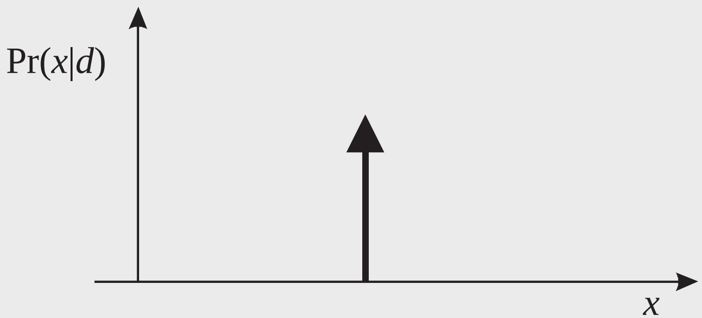
Posterior distribution for x conditional on d

$$\pi\,(x|d,m) \propto \mathrm{Pr}\,(d|x,m)\mathrm{Pr}\,(x|m) \qquad \text{(Bayes' rule)}$$

Posterior distribution is sole basis for inference

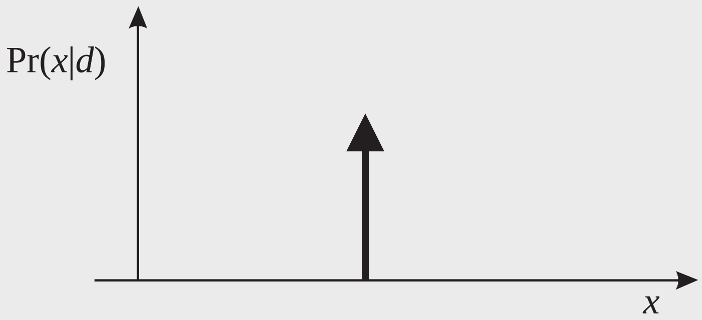
Often $$\pi(x|d,m) \propto \exp\left\{-\chi\,(d - A(x)) - \rho\,(x)\right\}$$

Interpreting the Posterior Distribution



Unique solution

Interpreting the Posterior Distribution



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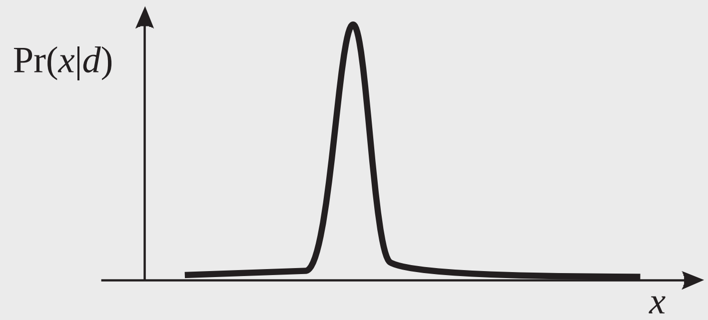
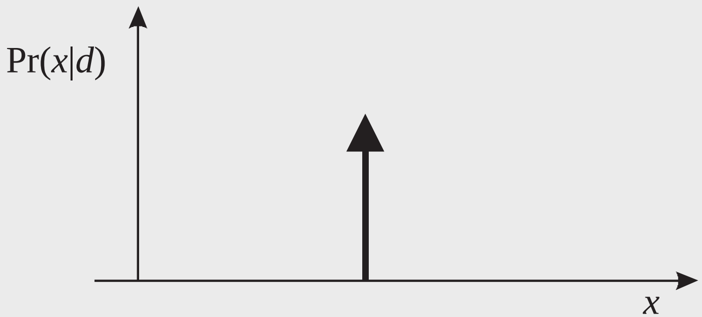
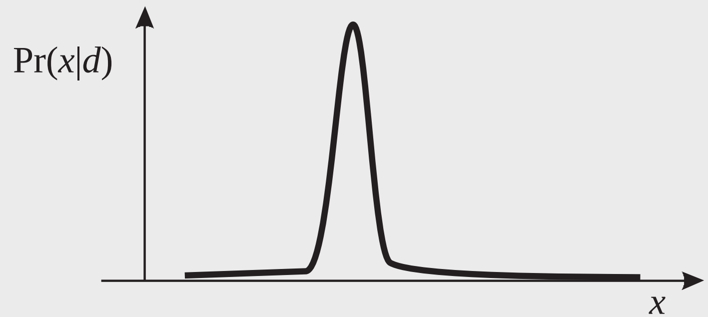


Solution localized

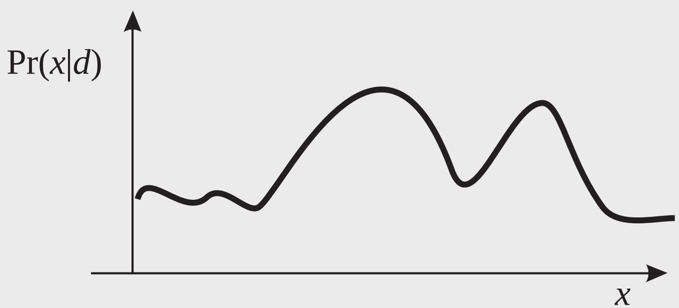
Interpreting the Posterior Distribution



Unique solution

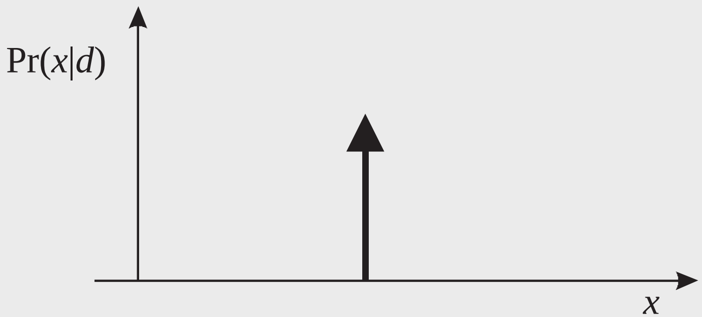


Solution localized



Solution not localized

Interpreting the Posterior Distribution



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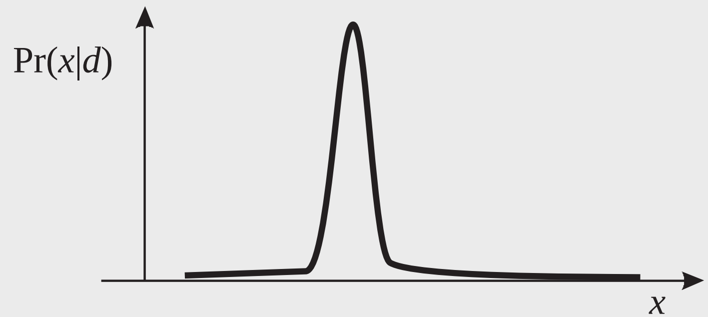

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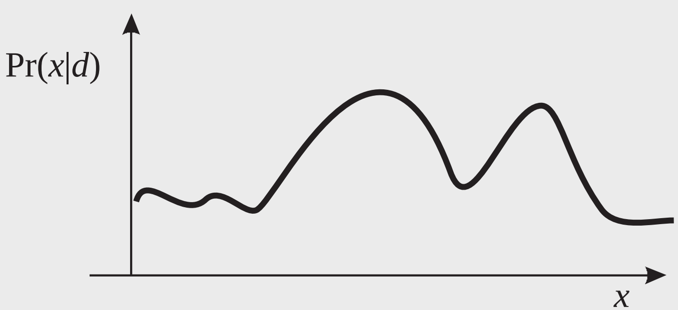

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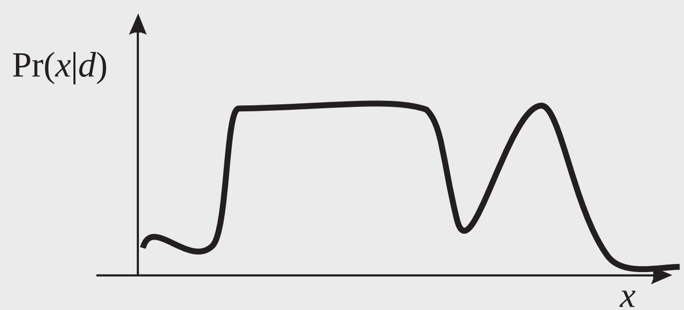


Multiple solutions

Solutions = Summary Statistics

Bayes' rule produces the posterior distribution $\pi(x|d)$ containing all information

Traditional Solutions - modes

$$\hat{x}_{\mathsf{MLE}} = \arg\max \Pr(d|x) \qquad \hat{x}_{\mathsf{MAP}} = \arg\max \pi(x|d)$$

e.g. Gaussian noise and prior: $\Pr(x) \propto \exp\left(-|x|^2/2\lambda^2\right)$

$$\hat{x}_{\mathsf{MAP}} = \arg\min |d - A(x)|^2 + \alpha |x|^2 \qquad \alpha = s^2/\lambda^2$$

- Tikhonov regularization, Kalman filtering, Backus-Gilbert, $\alpha = 0$ least-squares

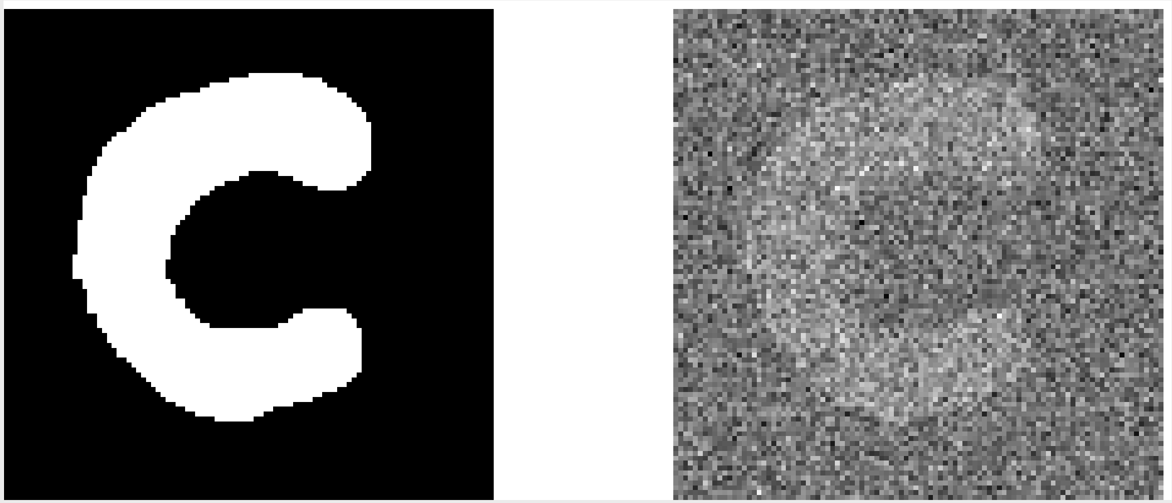
Inferential Solutions - expectations

$$\mathsf{E}[f(x)] = \int \pi(x|d)\, f(x)\, dx$$

E.g. if $f(x) =$ indicator function that image shows cancer

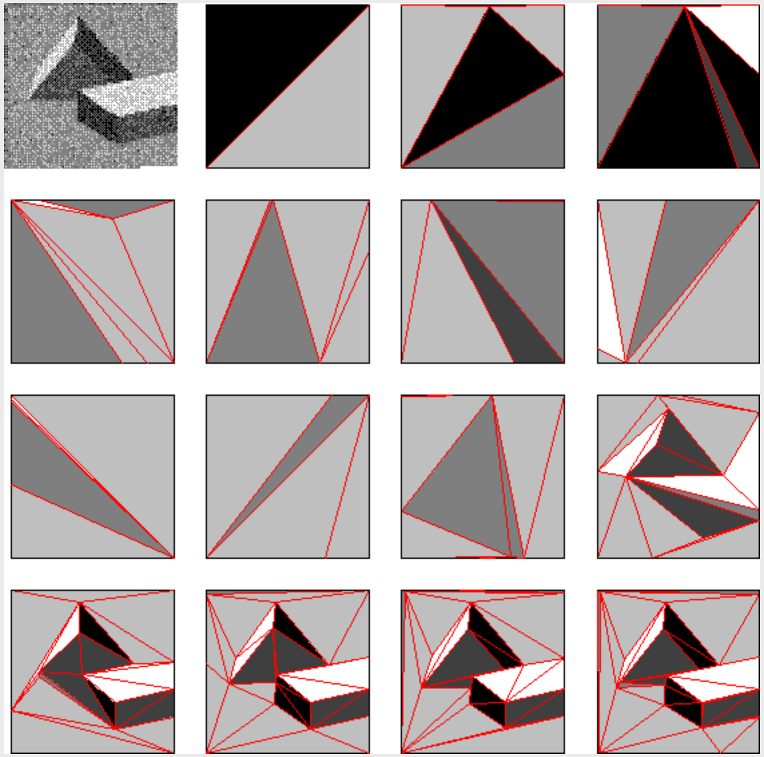
$\mathsf{E}[f(x)]$ is posterior probability (based on measurements, prior) that patient has cancer.

What questions are we trying to answer?



- “best” image
- If I know the image is binary (black and white) how many blobs are there?
- What is the area of the blob ?
- Does the blob have an inclusion (‘C’ or ‘O’)
- what is the cost of getting that decision wrong?

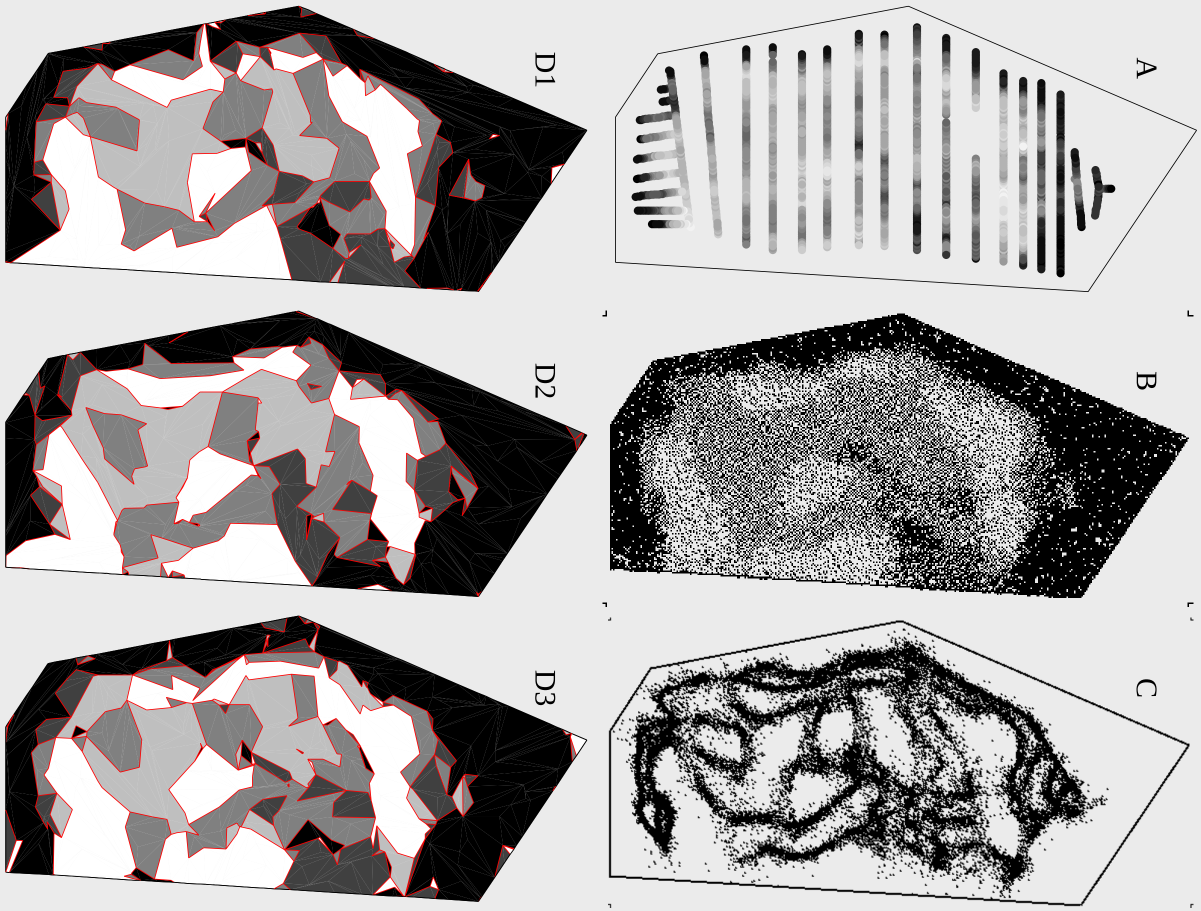
Coloured Continuum Triangulation



$$X = \bigcup_{i=0}^{\infty} \{[0,1] \times [0,1]\}^i \text{, coloured}$$

Geoff Nicholls, *Bayesian image analysis with Markov chain Monte Carlo and colored continuum triangulation models* JRSSB **60**:3 643-659 (1998)

Neolithic hill fort (Maori pa)



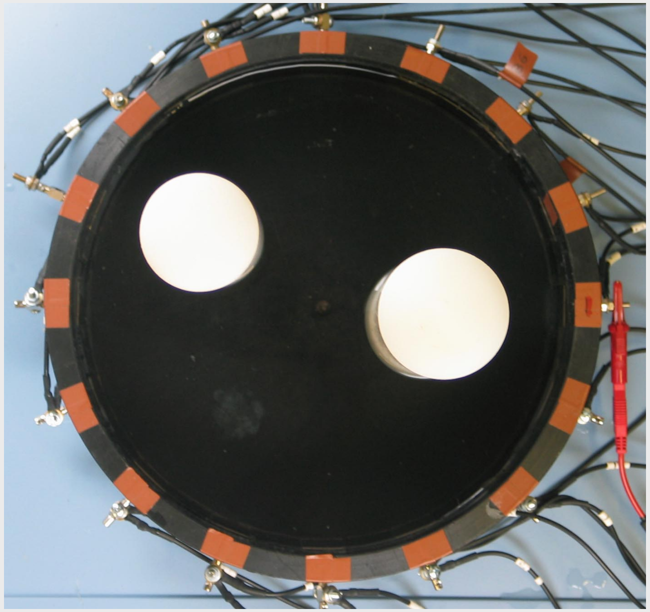
A) data, 1746 resistivity readings, (B) posterior mean resistivity, (C) posterior edge length density, (D1-3) samples from posterior

Electrical Impedance Tomography

For fixed current patterns $\{I\}$

$$A : \sigma \mapsto \{U\}$$

Simulate A by solving the BVP



$$\nabla \cdot \sigma \nabla u = 0$$

$$\int_{e_l} \sigma \frac{\partial u}{\partial n} dS = I_l$$

$$\sigma \frac{\partial u}{\partial n}\bigg|_{\partial\Omega \setminus \bigcup_l e_l} = 0$$

$$\left(u + z_l \sigma \frac{\partial u}{\partial n}\right)\bigg|_{e_l} = U_l$$

Posterior density

$$\pi(\sigma \mid V) \sim \exp\left\{-\left(\frac{1}{2}(V - U(\sigma))^{\mathrm{T}} C_n^{-1} (V - U(\sigma))\right)\right\} \pi_{\mathrm{pr}}(\sigma)$$

Gaussian smoothness prior

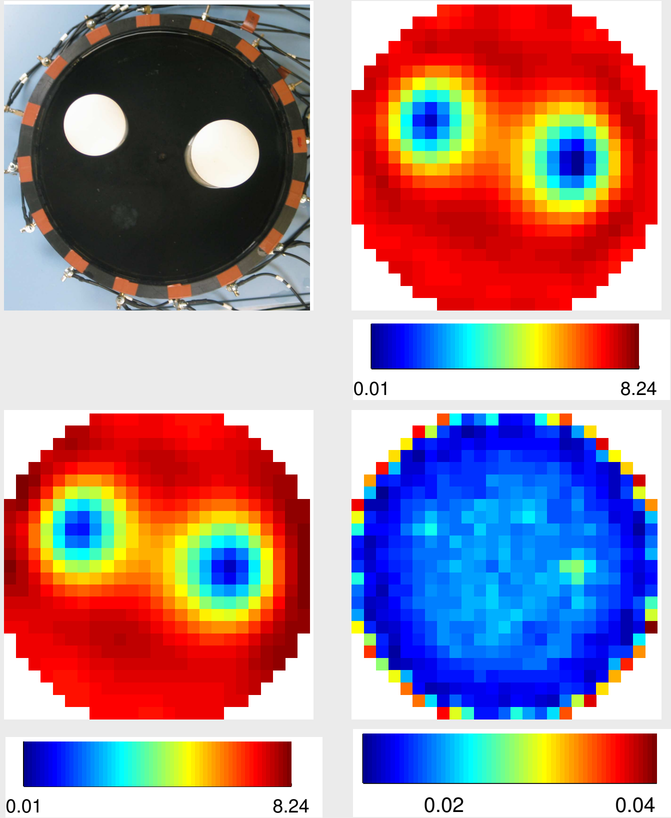


Figure 1: Results with the Gaussian smoothness MRF-prior. Top left: Photograph of the measurement setup. Top right: Maximum a posteriori estimate σ_{MAP} by the Gauss-Newton optimization algorithm. Bottom left and right: Posterior mean σ_{CM} and variance based on the MCMC simulation.

Kolehmainen, Fox and Nicholls, *MCMC Inversion of Measured EIT Data*, 200?

Material type prior – Nicholls, F 1998

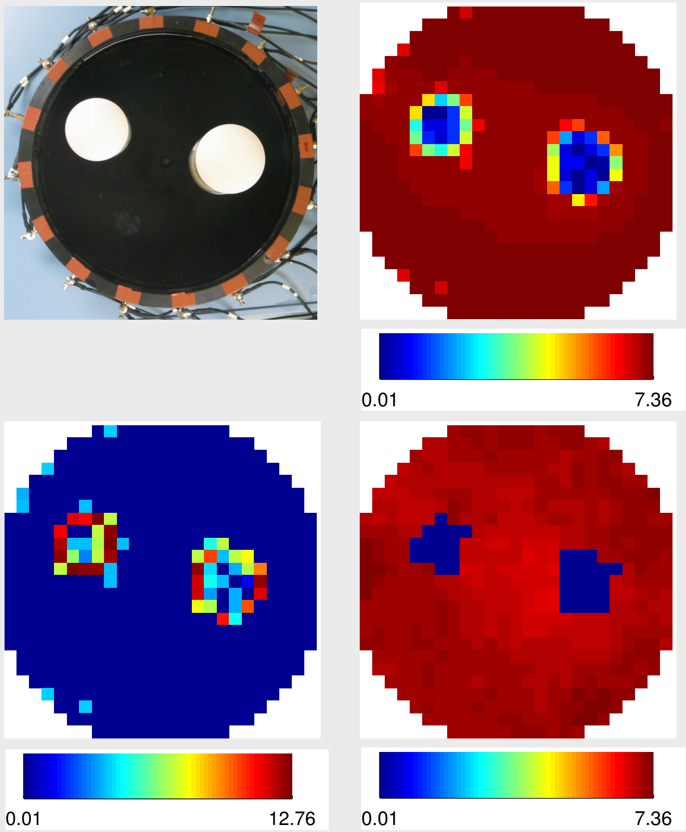
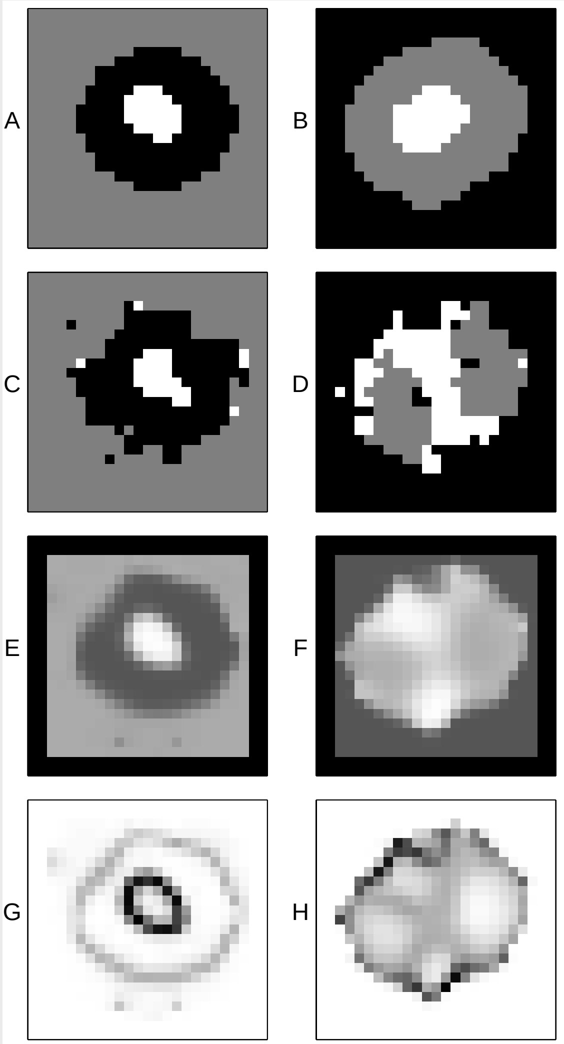


Figure 3: Results with the Material type MRF-prior. Top left: Photograph of the measurement setup. Top right: Posterior mean for the conductivity. Bottom left: Posterior variance of the conductivity. Bottom right: One sample from the posterior.

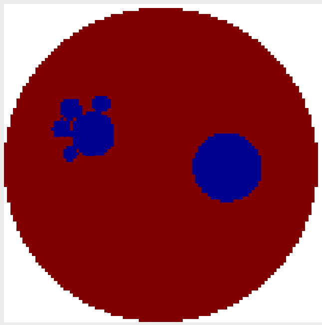
Kolehmainen F Nicholls *MCMC Inversion of Measured EIT Data*, 200?

Uncertainty due to shielding



Nicholls, F (1998)

Circular inclusions prior



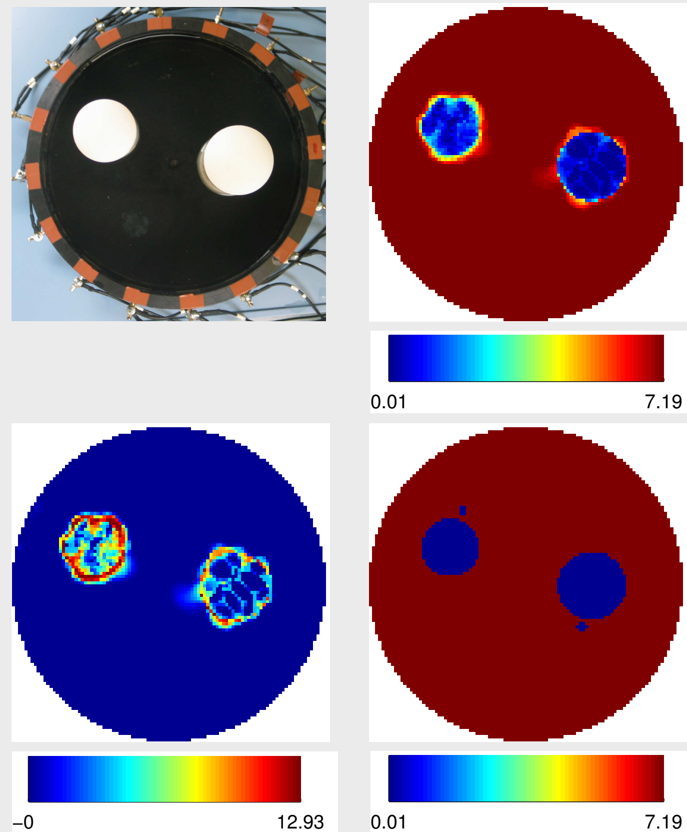
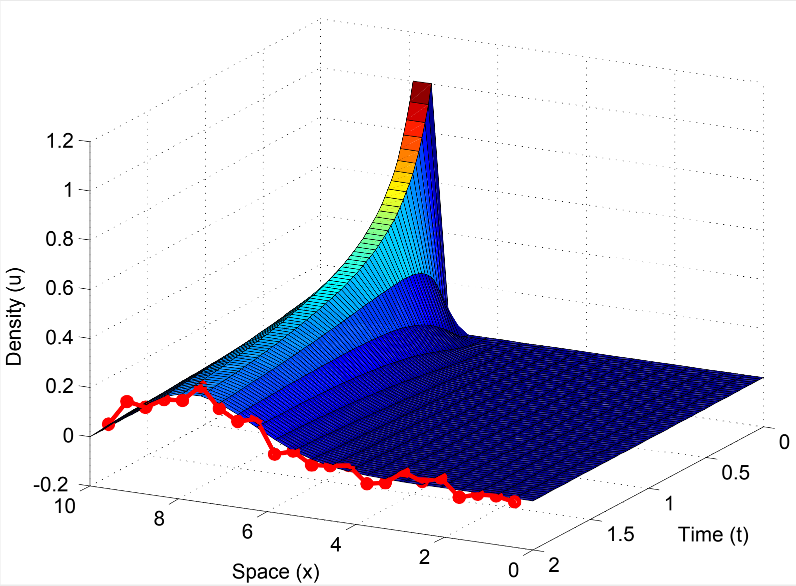


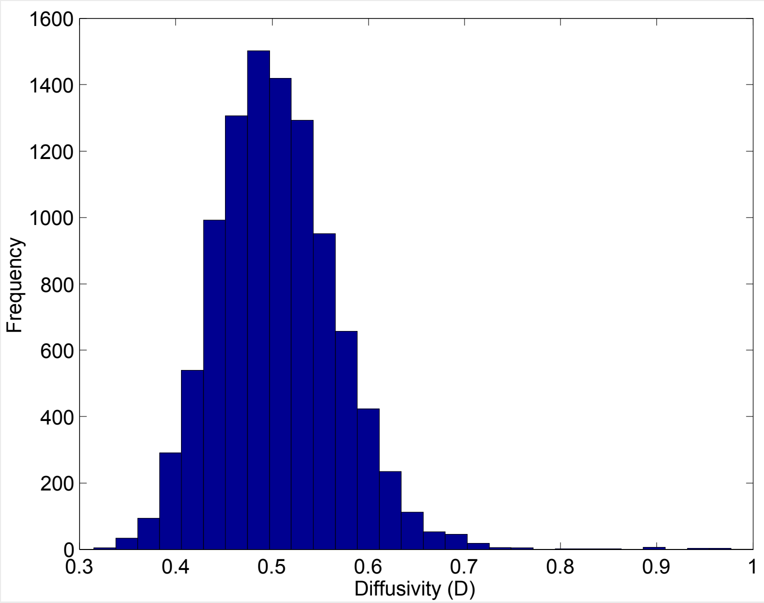
Figure 5: Results with the circle prior. Top left: Photograph of the measurement setup. Top right: Posterior mean for the conductivity. Bottom left: Posterior variance of the conductivity. Bottom right: Sample from the posterior.

Kolehmainen F Nicholls *MCMC Inversion of Measured EIT Data*, 200?

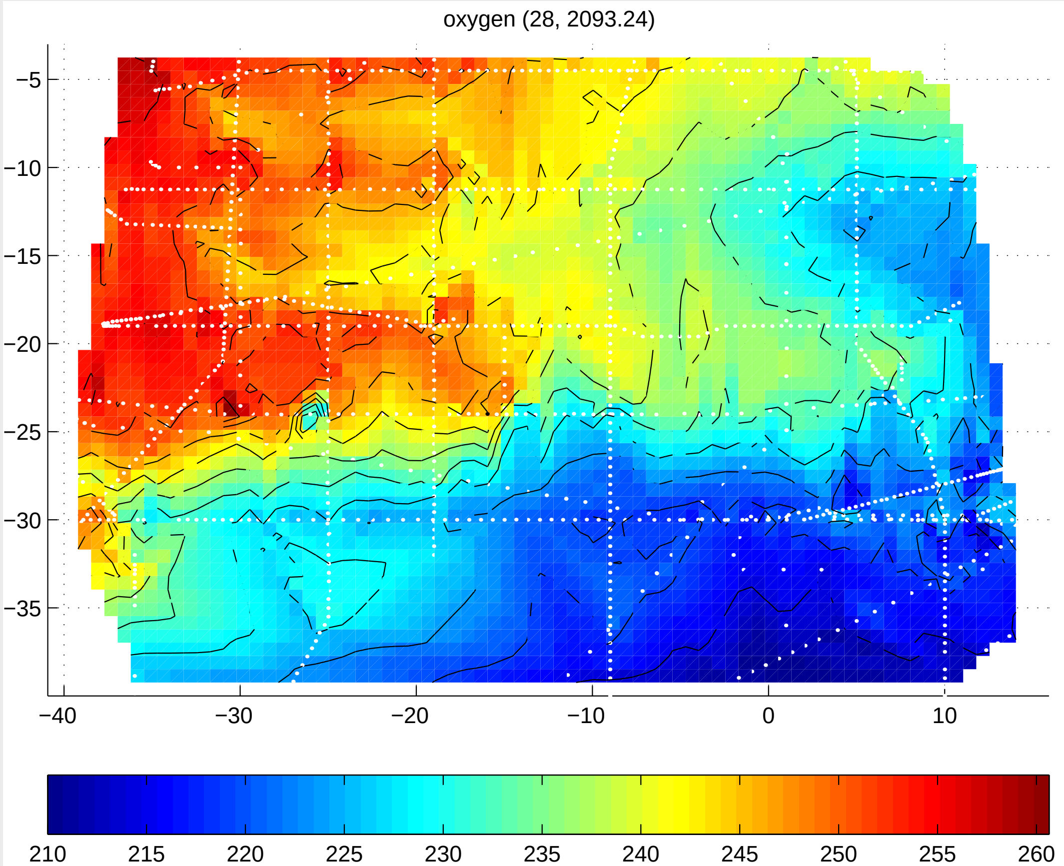
Estimation coefficient in a PDE :: diffusion

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

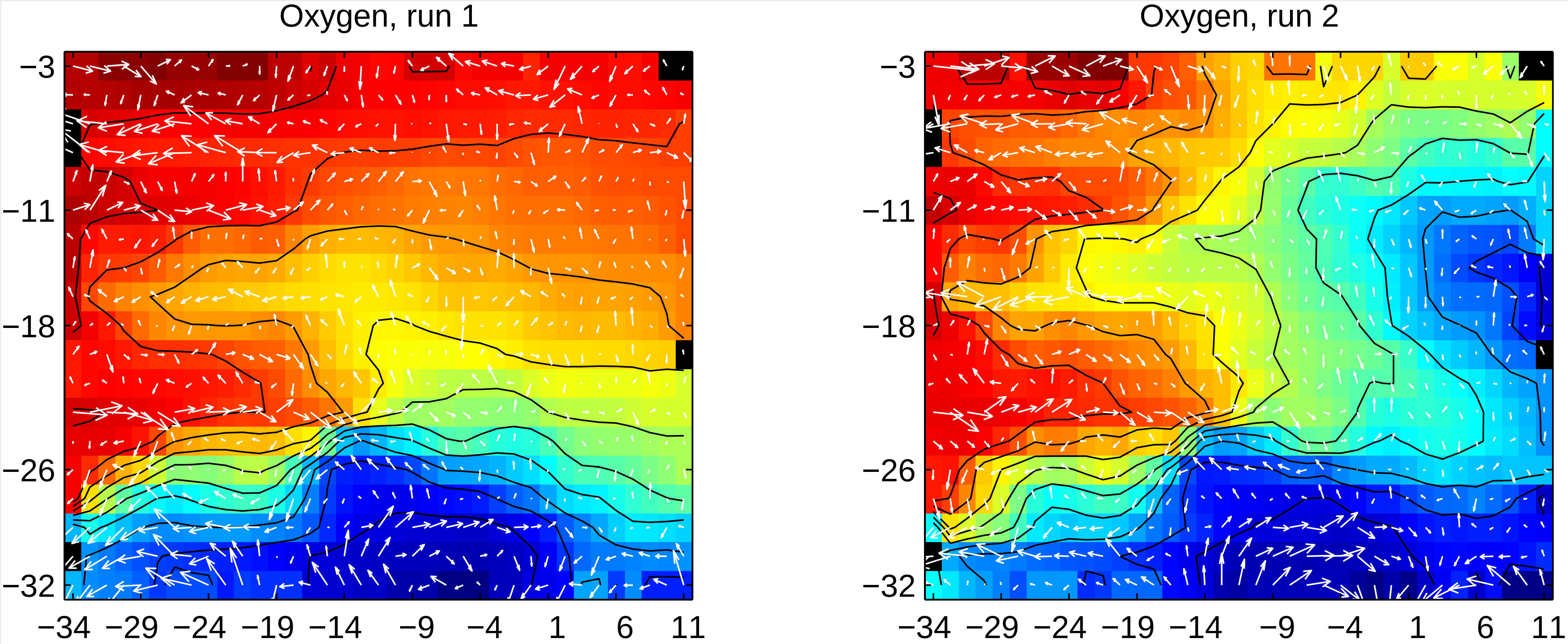




Oceanography :: abyssal advection

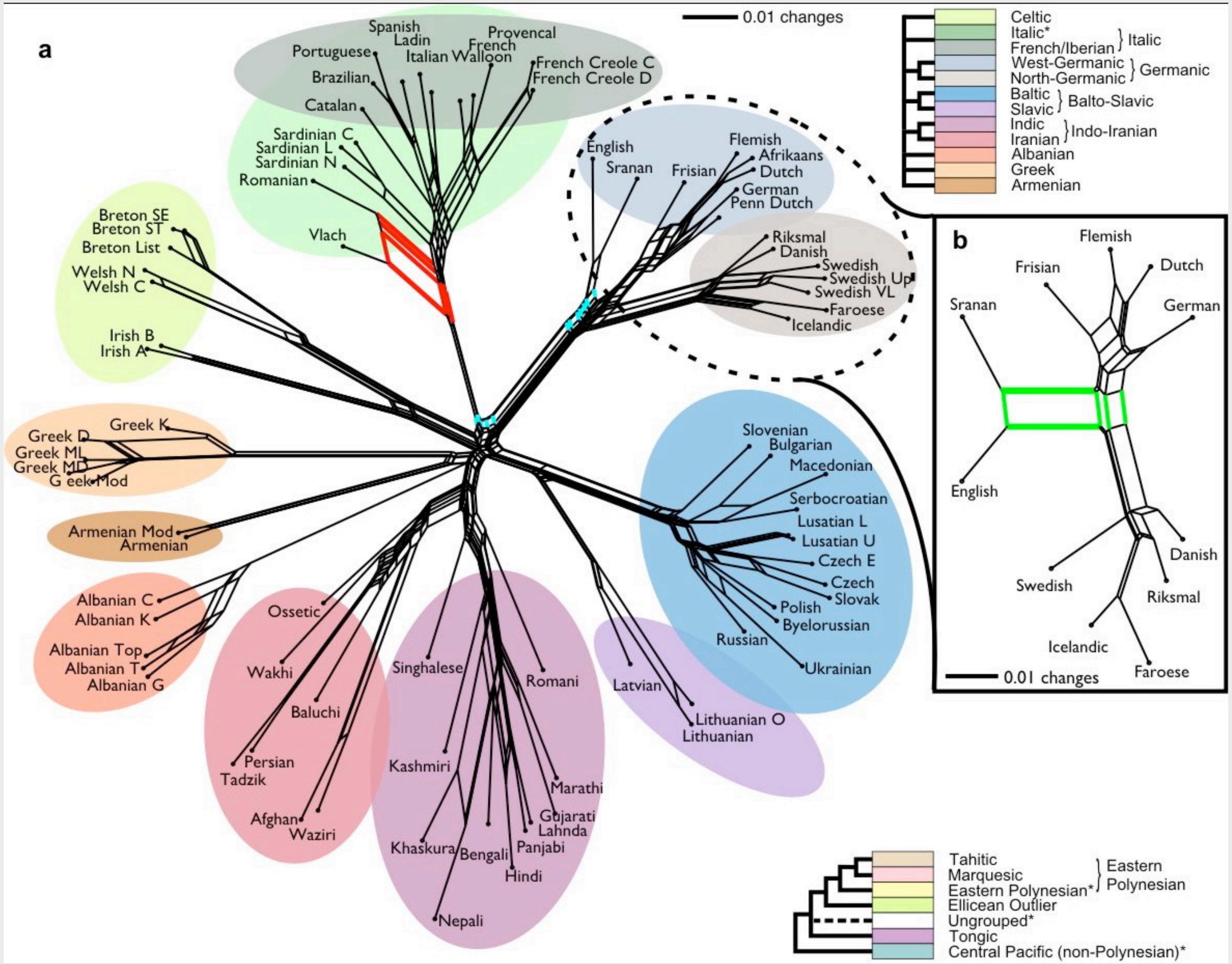


Oceanography :: 2 samples



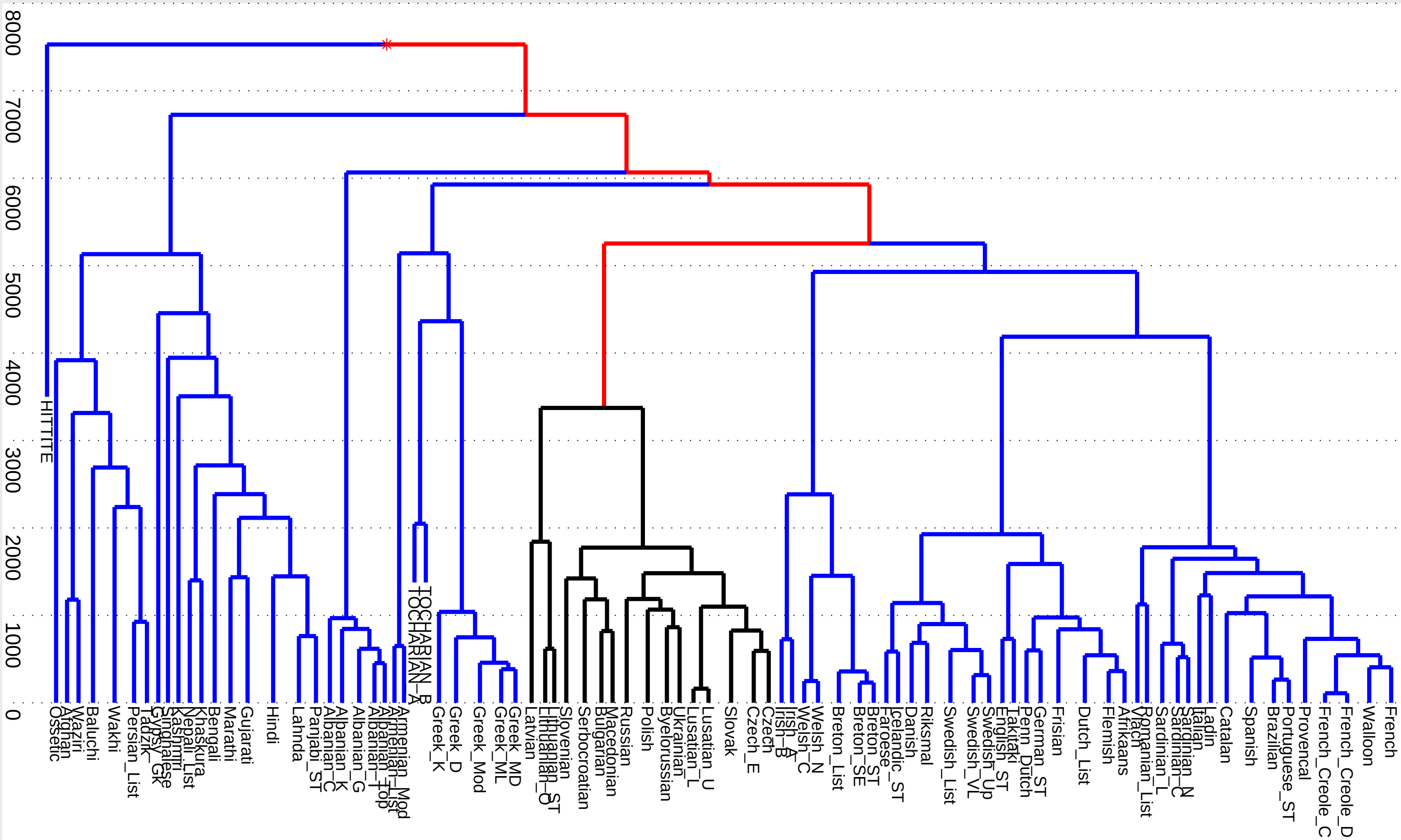
McKeague Nicholls Speer Herbei, Statistical Inversion of South Atlantic Circulation in an Abyssal Neutral Density Layer, *Journal of Marine Research* 2005

Tree/Graph Model of Language Evolution



Bryant, Gray (2006)

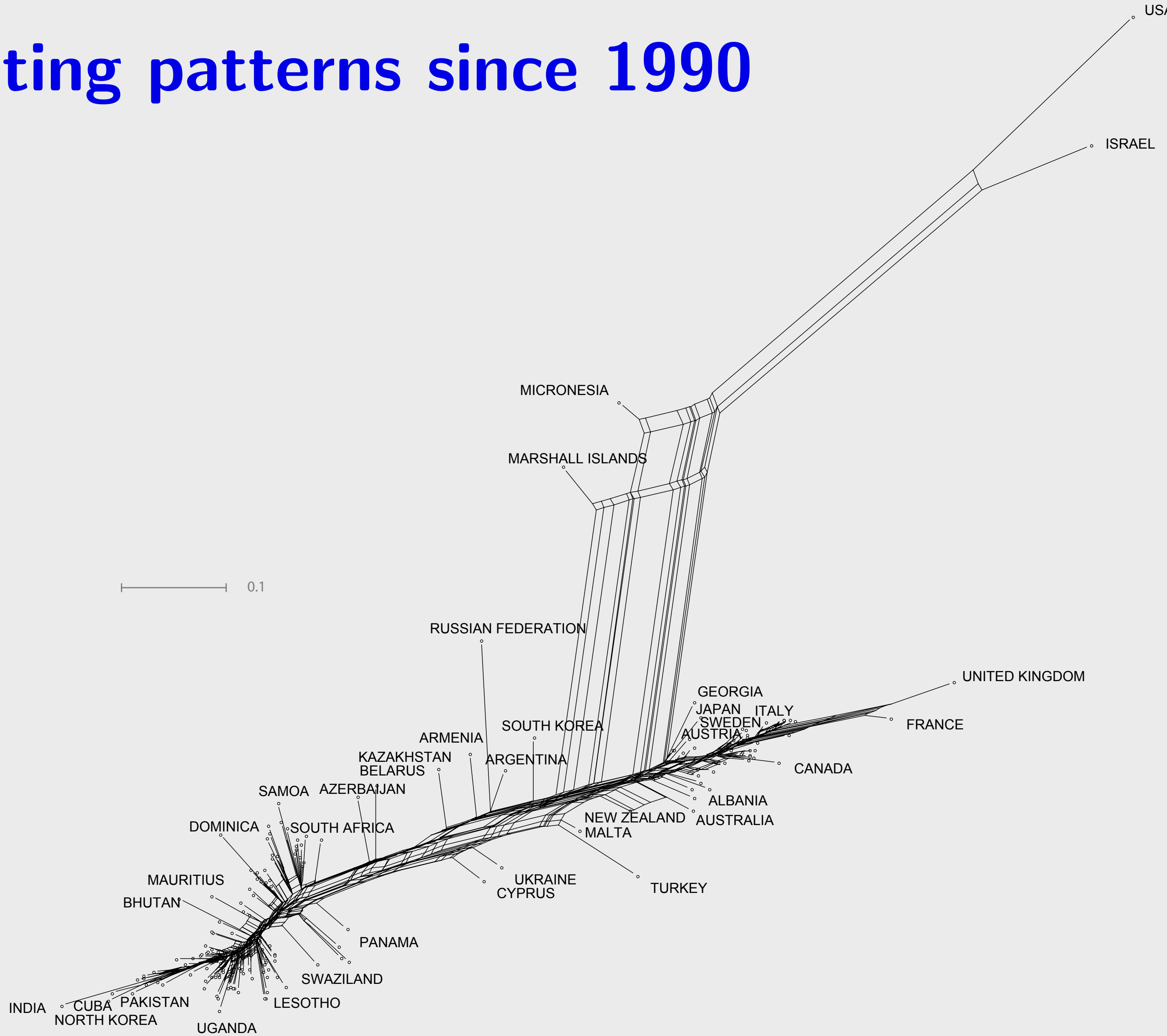
Stochastic Dollo Model



θ =branching rate, λ =cognate birth rate, μ =per capita death rate. Exact integral over θ, λ, MCMC for μ and graphs.

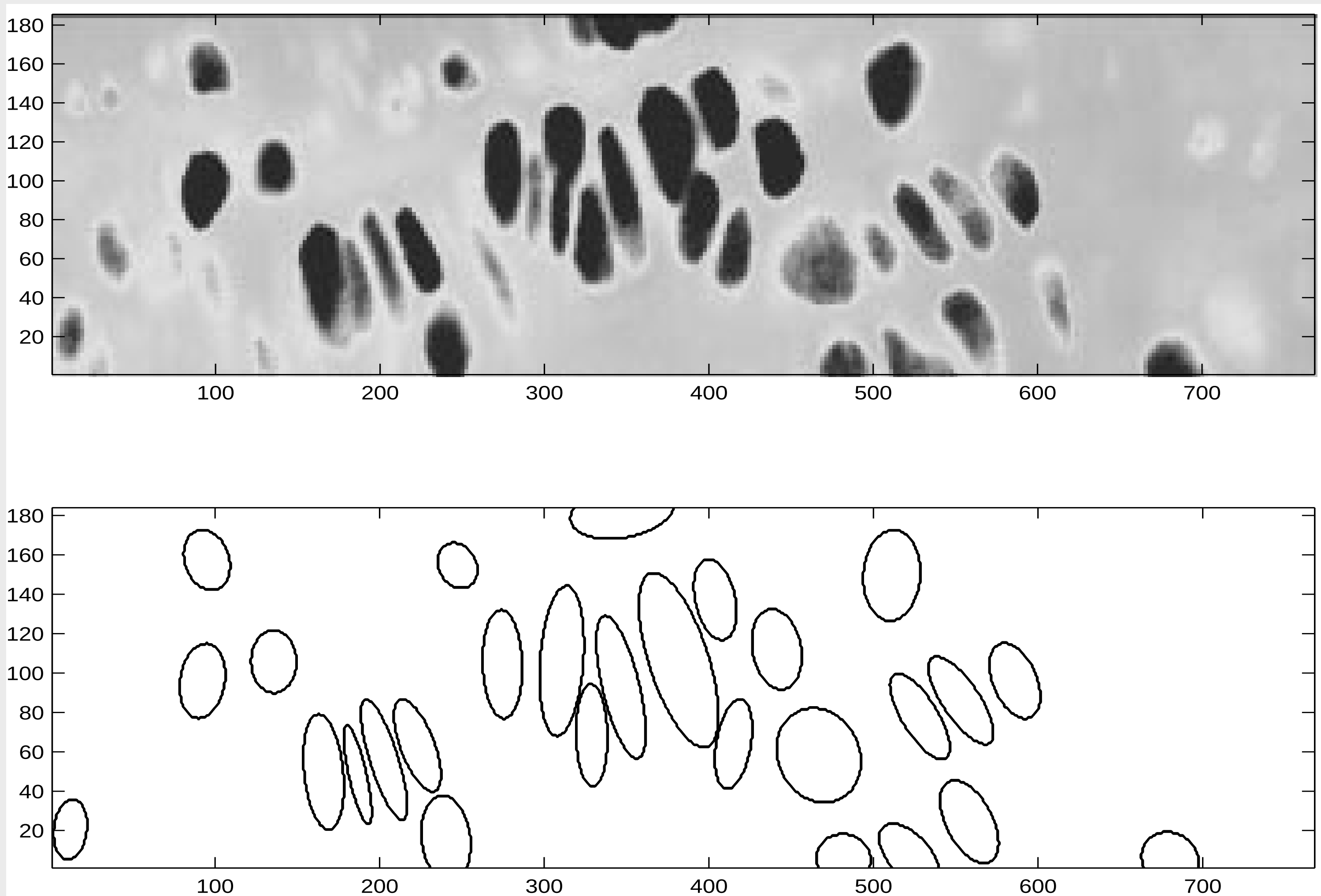
Nicholls, Gray (2002)

U.N. voting patterns since 1990



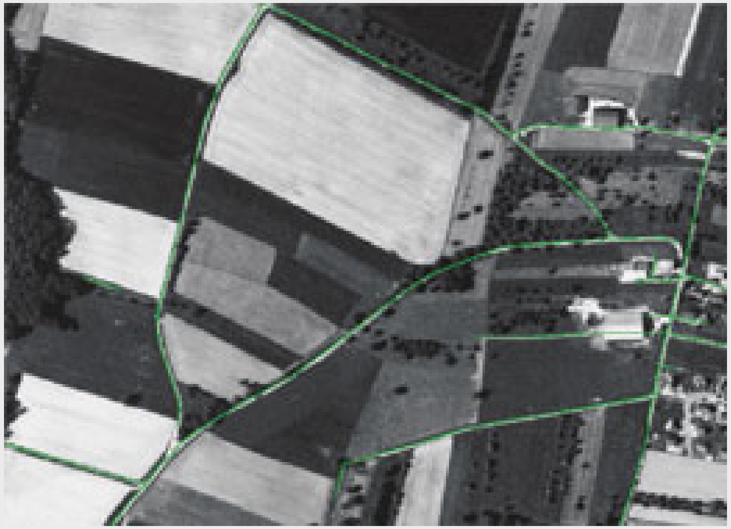
thanks to David Bryant

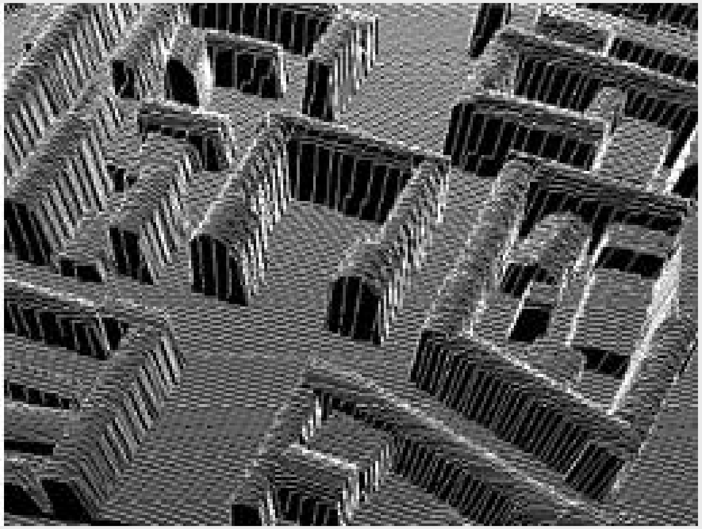
Marked Point Process



Fahimah Al-Awadhi, Christopher Jennison, Merrilee Hurn (2003)

Marked Point Process (cont)





Josiane Zerubia, Xavier Descombes, C. Lacoste, M. Ortner, R. Stoica (2000, 2003)

Details :: Markov chain Monte Carlo

- Monte Carlo integration: If $\{X_t, t = 1, 2, \ldots, n\}$ are sampled from $\pi(x)$

$$\mathsf{E}[f(x)] \approx \frac{1}{n} \sum_{t=1}^{n} f(X_t)$$

- Markov chain: Generate $\{X_t\}_{t=0}^{\infty}$ as a Markov chain of random variables $X_t \in X$, with a t-step distribution $\Pr(X_t = x | X_0 = x^{(0)})$ that tends to $\pi(x)$, as $t \to \infty$.

Metropolis-Hastings algorithm

1. given state x_t at time t generate candidate state x' from a proposal distribution $q(.|x_t)$
2. With probability $\alpha(x_t \to x') = \min\left(1, \frac{\pi(x')q(x_t|x')}{\pi(x_t)q(x'|x_t)}\right)$
   set $X_{t+1} = x'$ otherwise $X_{t+1} = x_t$
3. Repeat

$q(.|x_t)$ can be any distribution that ensures the chain is irreducible and aperiodic.

Conclusions

1. Inferential formulation quantifies uncertainty in unknown x
2. Bayesian methods give a machinery for combining uncertainties, forward modelling, expert knowledge, cost of decisions, etc
3. Provide posterior uncertainties for given data (cf. CRLB)
4. In principle all desired computations possible using MCMC
5. These methods solve substantial problem in tomography, image classification, economics, biology, history, ...
6. Lots of outstanding research issues