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Prior Modeling and Posterior Sampling in Conductivity Imaging

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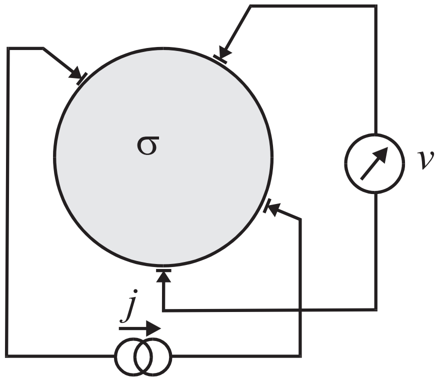
Overview

- Conductivity Imaging
- Statistical model for inverse problems
- Markov chain Monte Carlo
- Conductivity Imaging using various prior models
- Mid- and high-level models

Conductivity Imaging Measurements



- Electrodes at $x_1, x_2, \cdots, x_E$
- Assert currents at electrodes $j = \left(j\left(x_1\right), j\left(x_2\right), \cdots, j\left(x_E\right)\right)^T$
- Measure voltages $v = \left(\phi\left(x_1\right), \phi\left(x_2\right), \cdots, \phi\left(x_E\right)\right)^T$.

Unknown $\sigma\left(x\right)$ related to measurements via Neumann BVP

$$\begin{aligned} \nabla \cdot \sigma\left(x\right) \nabla \phi\left(x\right) &= 0 && x \in \Omega \\ \sigma\left(x\right) \frac{\partial \phi\left(x\right)}{\partial n\left(x\right)} &= j\left(x\right) && x \in \partial\Omega \end{aligned}$$

- Set of measurements is current-voltage pairs

$$\left\{j^n, v^n\right\}_{n=1}^{N}$$

Inverse problem is to find σ from these measurements (non linear)

Green's Functions

Unknown image $\sigma(x)$ related to measurements via Neuman BVP:

$$\begin{aligned} \nabla \cdot \sigma(x) \nabla \phi(x) &= s(x) && x \in \Omega \\ \sigma(x) \frac{\partial \phi(x)}{\partial n(x)} &= j(x) && x \in \partial\Omega \end{aligned}$$

plus potential reference

If σ is compiled into a certain matrix, measurements correspond to certain elements of the inverse.

Neuman Green's function $g(x|\xi)$:

$$\begin{aligned} \nabla \cdot \sigma(x) \nabla g(x|\xi) &= \delta(x-\xi) && \forall x \in \Omega \\ \sigma(x) \frac{\partial g(x|\xi)}{\partial n(x)} &= \tfrac{1}{|\partial\Omega|} && \forall x \in \partial\Omega \\ \textstyle\int_{\partial\Omega} g(x|\xi)\, dl(x) &= 0 \end{aligned}$$

$$g(x|\xi) = g(\xi|x).$$

Solutions to BVP:

$$\phi(x) = \int_{\Omega} g(x|\xi)\, s(\xi)\, d\xi + \int_{\partial\Omega} g(x|\xi)\, j(\xi)\, dl(\xi)$$

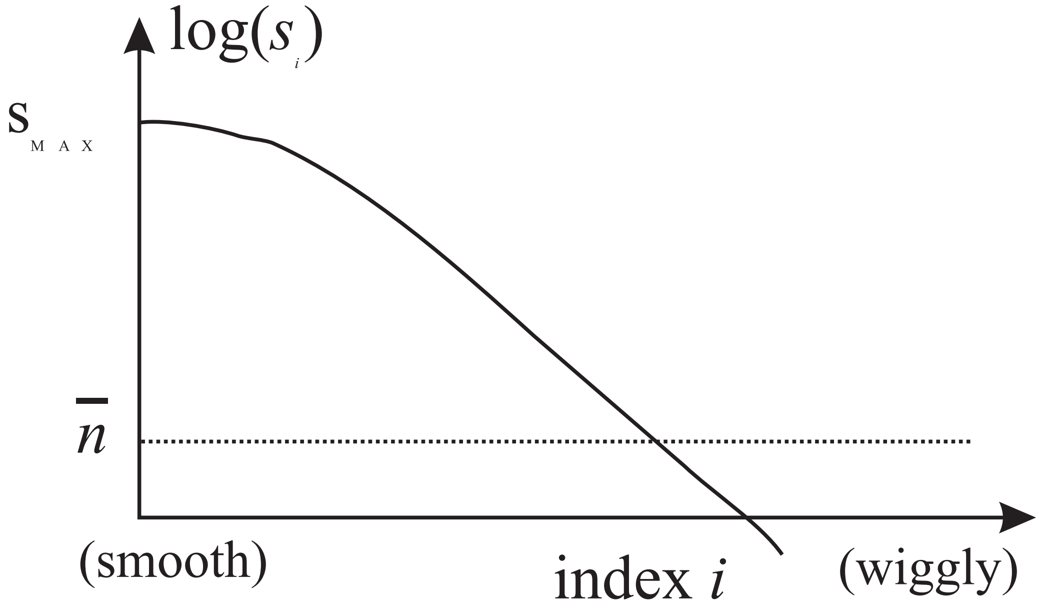
- $\Gamma_\sigma : j \to \phi$ (Neumann to Dirichlet map) is linear.
- $\sigma \to \Gamma_\sigma$ is not linear.
- Inverse problem: Measure Γ_σ, want σ.

Properties of the Inverse Problem

- $\sigma \mapsto \Gamma_\sigma$ is invertible for $\sigma \in C^\infty\left(\Omega \subset \mathbb{R}^2\right)$

  $0 < \sigma_{\mathsf{min}} \leq \sigma \leq \sigma_{\mathsf{max}} < \infty$

- Fréchet derivative $\dfrac{\partial \Gamma_\sigma}{\partial \sigma}$ has singular-values that decrease $\sim$ geometrically



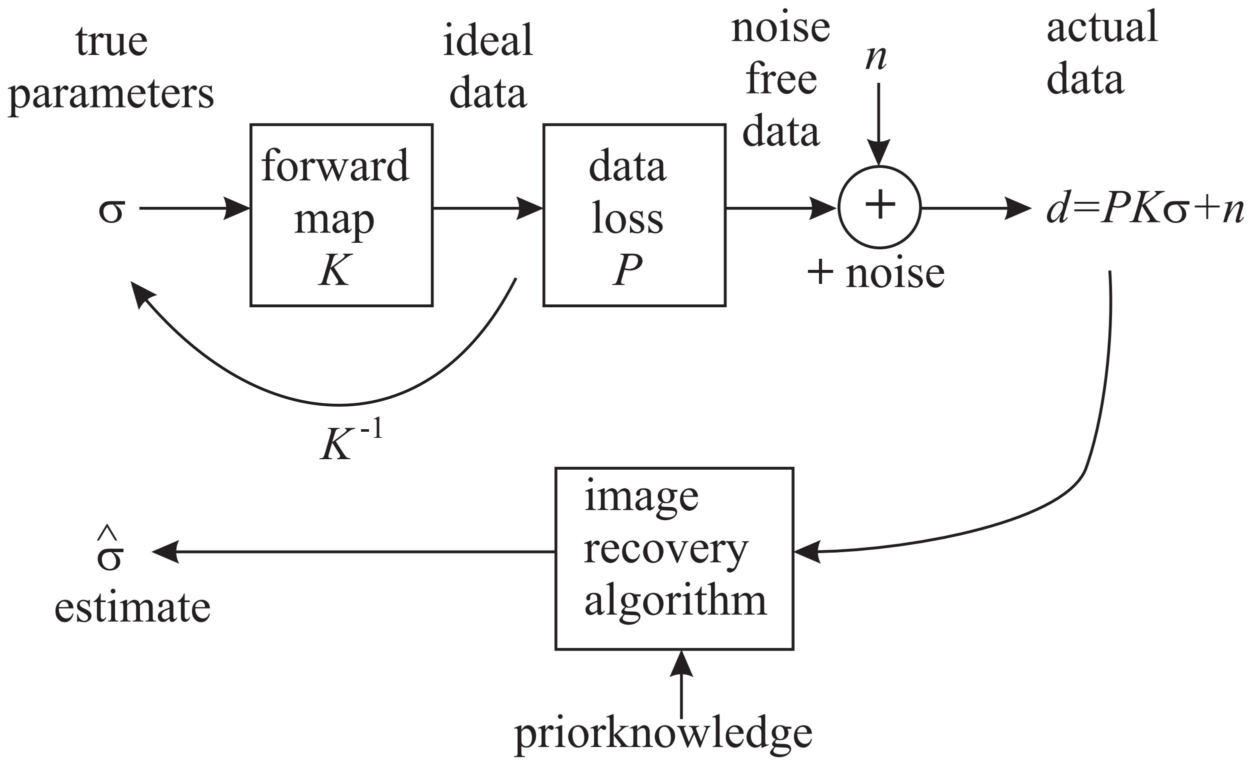
$$\hat{\sigma}_i = \frac{\sigma_i s_i + n_i}{s_i} = \sigma_i + \frac{n_i}{s_i}$$

roughly, data measured when $\dfrac{s_{\mathsf{max}}}{s_i} \leq \mathsf{SNR}$

- Inverse discontinuous

- Measurements cannot define image uniquely

Statistical Model for Imaging



If $n \sim f_N(n)$ then $d \sim f_{D|\Sigma}(d|\sigma) = f_N(d - PK\sigma)$

Given measurements v, the likelihood for σ is

$$L_d(\sigma) \equiv \Pr(d|\sigma) = f_N(d - PK\sigma)$$

Posterior distribution for σ conditional on v

$$\Pr(\sigma|d) = \frac{f_{D|\Sigma}(d|\sigma)\Pr(\sigma)}{\sum_{\sigma \in \Sigma_\Omega} f_{D|\Sigma}(d|\sigma)} \qquad \text{(Bayes rule)}$$

In **subjectivist** formulation, prior and posterior distributions for σ are quantified representations of our state of knowledge

Summary Statistics

All information contained in posterior distribution $\Pr(\sigma|v)$

"Answers" are expectations over the posterior

$$\mathsf{E}[f(\sigma)] = \int \Pr(\sigma|v)\, f(\sigma)\, d\sigma$$

Decision based on utility function

		σ F	σ T
$\hat{\sigma}$	F	c_{TT}	c_{TF}
	T	c_{FT}	c_{FF}

Image is an intermediate step

Nuisance parameters

Data depends on image σ and parameters θ

$$\Pr(\sigma|v) = \int_{\Theta} \Pr(\sigma, \theta|v)\, d\theta$$

e.g. true currents or voltages

Monte Carlo Integration

$$I = \mathsf{E}\left[f\left(\sigma\right)\right] = \int_{\Sigma} \pi\left(\sigma\right) f\left(\sigma\right)\, d\sigma$$

Simple case

Draw $\sigma^{(1)}, \ldots, \sigma^{(m)}$ uniformly on Σ

$$\hat{I} = \frac{1}{m}\left\{f\left(\sigma^{(1)}\right) + \cdots + f\left(\sigma^{(m)}\right)\right\}$$

$\hat{I} = I + O\left(m^{-1/2}\right)$

c.f. $\left\{\sigma^{(i)}\right\}$ regular grid on Σ, $\hat{I} = I + O\left(m^{-1}\right)$

Importance sampling

- $\sigma^{(1)}, \ldots, \sigma^{(m)}$ drawn from $g(\cdot)$
- Importance weight $w^{(i)} = \pi\left(\sigma^{(i)}\right)/g\left(\sigma^{(i)}\right)$
- $$\hat{I} = \frac{\left\{w^{(1)} f\left(\sigma^{(1)}\right) + \cdots + w^{(m)} f\left(\sigma^{(m)}\right)\right\}}{\left\{w^{(1)} + \cdots + w^{(m)}\right\}}$$

  c.f. unbiased estimate $\hat{I} = \frac{1}{m}\sum_i w^{(i)} f\left(\sigma^{(i)}\right)$

Only need $\pi(\cdot)/g(\cdot)$ up to multiplicative constant

Choose $g(\cdot)$ close to shape of $\pi(\cdot)/f(\cdot)$

Bayesian Formulation for Conductivity Imaging

	current in Ω	potential in Ω	voltage electrode	current electrode	conductivity
r.v.	R	Φ	V	J	Σ
value	ρ	ϕ	v	j	σ

Joint Posterior

$$\begin{aligned} &\Pr\left\{\sigma, \phi^n, \rho^n \middle| \left\{j^n, v^n\right\}\right\} \\ &\quad = \Pr\left\{\left\{j^n, v^n\right\}\middle|\sigma, \phi^n, \rho^n\right\} \times \Pr\left\{\sigma, \phi^n, \rho^n\right\} \end{aligned}$$

$\phi = \Gamma_\sigma\left(\rho|_{\partial\Omega}\right)$ and $\rho = -\sigma\nabla\phi$

$$\Pr\left\{\sigma, \phi^n, \rho^n\right\} = \Pr\left\{\sigma, \rho^n\right\}$$

Stipulate $\Pr\{\sigma\}$ only in examples – usually a MRF

$$\begin{aligned} L\left(\sigma, \phi^n, \rho^n\right) &= \Pr\left\{\left\{j^n, v^n\right\}\middle|\sigma, \phi^n, \rho^n\right\} \\ &= \Pr\left\{\left\{v^n\right\}\middle|\phi^n\left(\sigma, \rho^n\right)\right\} \times \Pr\left\{\left\{j^n\right\}\middle|\rho^n\right\} \end{aligned}$$

Errors i.i.d.

$$L\left(\sigma, \phi^n, \rho^n\right) = \Pi_{n=1}^{N} \Pr\left\{v^n \middle| \Gamma_\sigma\left(\rho^n|_{\partial\Omega}\right)\right\} \times \Pr\left\{j^n \middle| \rho^n\right\}.$$

Markov chain Monte Carlo

- Monte Carlo integration

  If $\{X_t, t = 1, 2, \ldots, n\}$ are sampled from $\Pr(\sigma|v)$

$$\mathsf{E}[f(\sigma)] \approx \frac{1}{n}\sum_{t=1}^{n} f(X_t)$$

- Markov chain

  Generate $\{X_t\}_{t=0}^{\infty}$ as a Markov chain of random variables $X_t \in \Sigma_\Omega$, with a t-step distribution $\Pr(X_t = \sigma | X_0 = \sigma^{(0)})$ that tends to $\Pr(\sigma|v)$, as $t \to \infty$.

Metopolis-Hastings algorithm

1. given state $\sigma_t$at time t generate candidate state σ' from a proposal distribution $q(.|\sigma_t)$
2. Accept candidate with probability

$$\alpha(X|Y) = \min\left(1, \frac{\Pr(Y|v)q(X|Y)}{\Pr(X|v)q(Y|X)}\right)$$

3. If accepted, $X_{t+1} = \sigma'$ otherwise $X_{t+1} = \sigma_t$
4. Repeat

$q(.|\sigma_t)$ can be any distribution that ensures the chain is:

– irreducible

– aperiodic

Three-Move Metropolis Hastings

Choose one of 3 moves with probability ζ_p, $p = 1, 2, 3$

Transition probabilities $\Pr^{(p)}$ reversible w.r.t. $\Pr(\sigma|v)$

$$\begin{aligned}\Pr(X_{t+1} &= \sigma_{t+1}|X_t = \sigma_t) \\ &= \sum_{p=1}^{3} \zeta_p \Pr{}^{(p)}(X_{t+1} = \sigma_{t+1}|X_t = \sigma_t).\end{aligned}$$

If at least one of the moves is irreducible on Σ_Ω, then the equilibrium distribution is $\Pr(\sigma|v)$.

A pixel n is a *near-neighbour* of pixel m if their lattice distance is less than $\sqrt{8}$.

An *update-edge* is a pair of near-neighbouring pixels of unequal conductivity. ($\mathcal{N}^*(\sigma)$, $\mathcal{N}^*_m(\sigma)$)

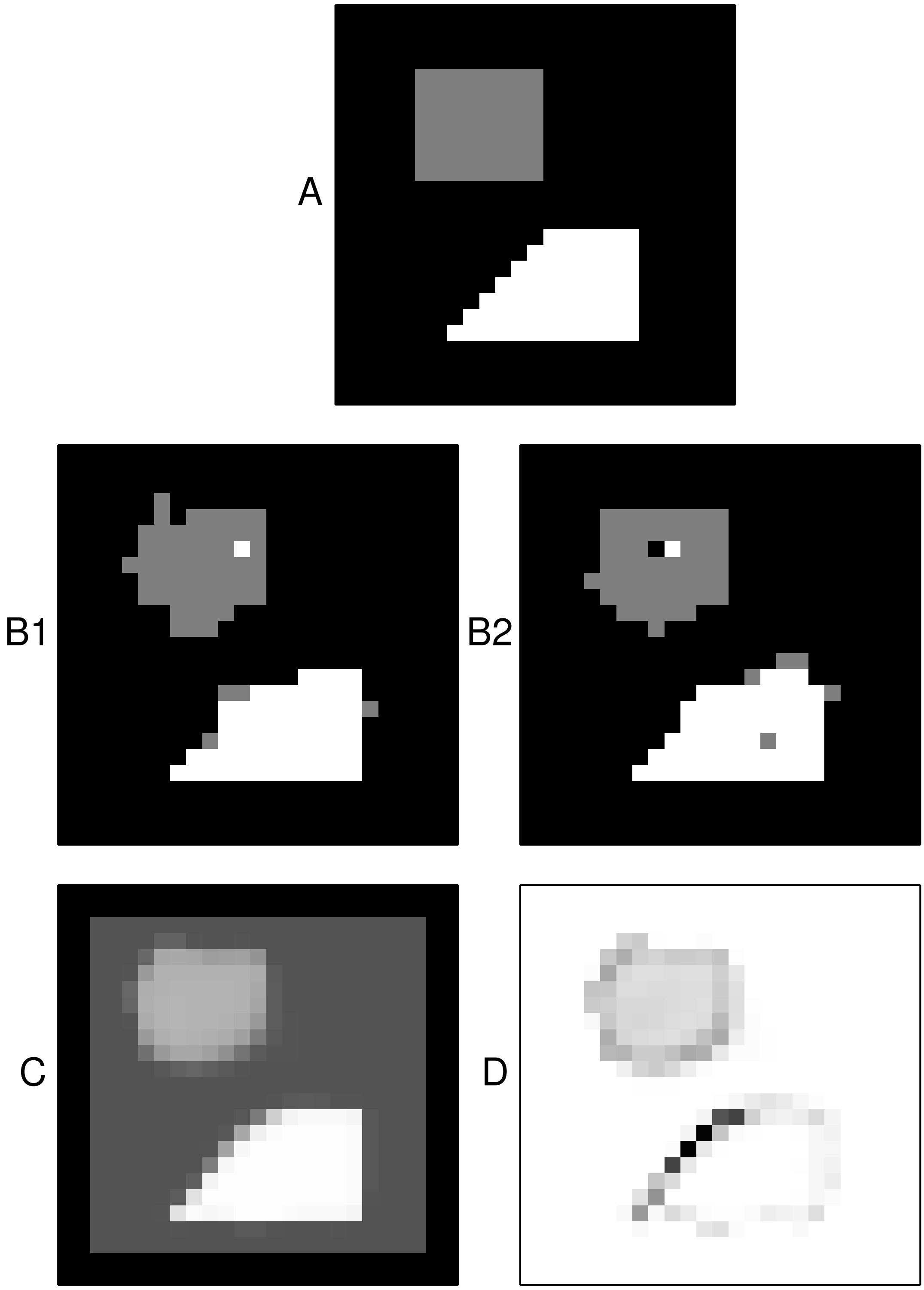
Move 1 *Flip a pixel.* Select a pixel m at random and assign σ_m a new conductivity σ'_m chosen uniformly at random from the other $\mathcal{C} - 1$ conductivity values.

Move 2 *Flip a pixel near a conductivity boundary.* Pick an update-edge at random from $\mathcal{N}^*(\sigma)$. Pick one of the two pixels in that edge at random, pixel m say. Proceed as in Move 1.

Move 3 *Swap conductivities at a pair of pixels.* Pick an update-edge at random from $\mathcal{N}^*(\sigma)$. Set $\sigma'_m = \sigma_n$ and $\sigma'_n = \sigma_m$.

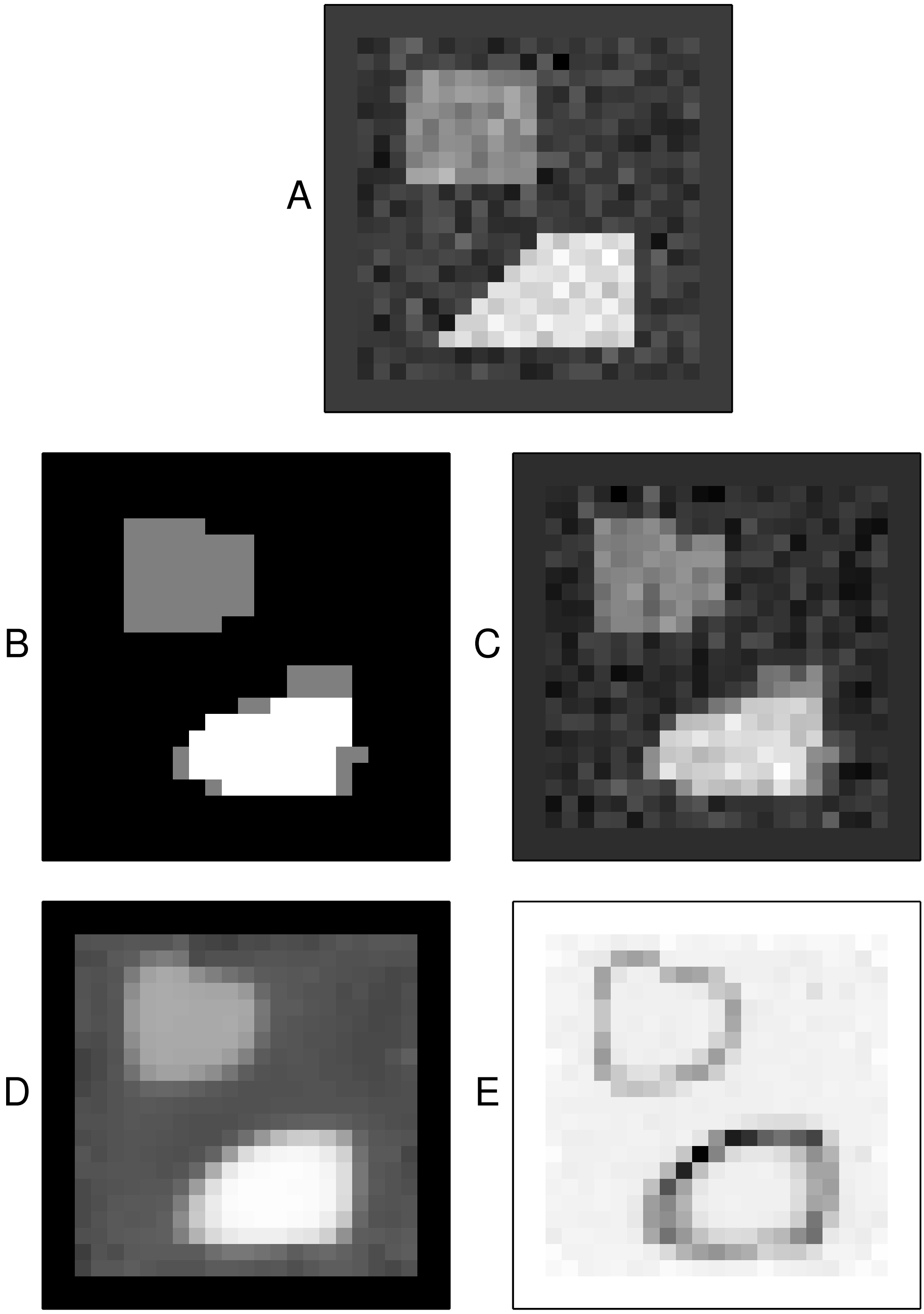
Experiment 1

(discrete variables – three conductivity levels)



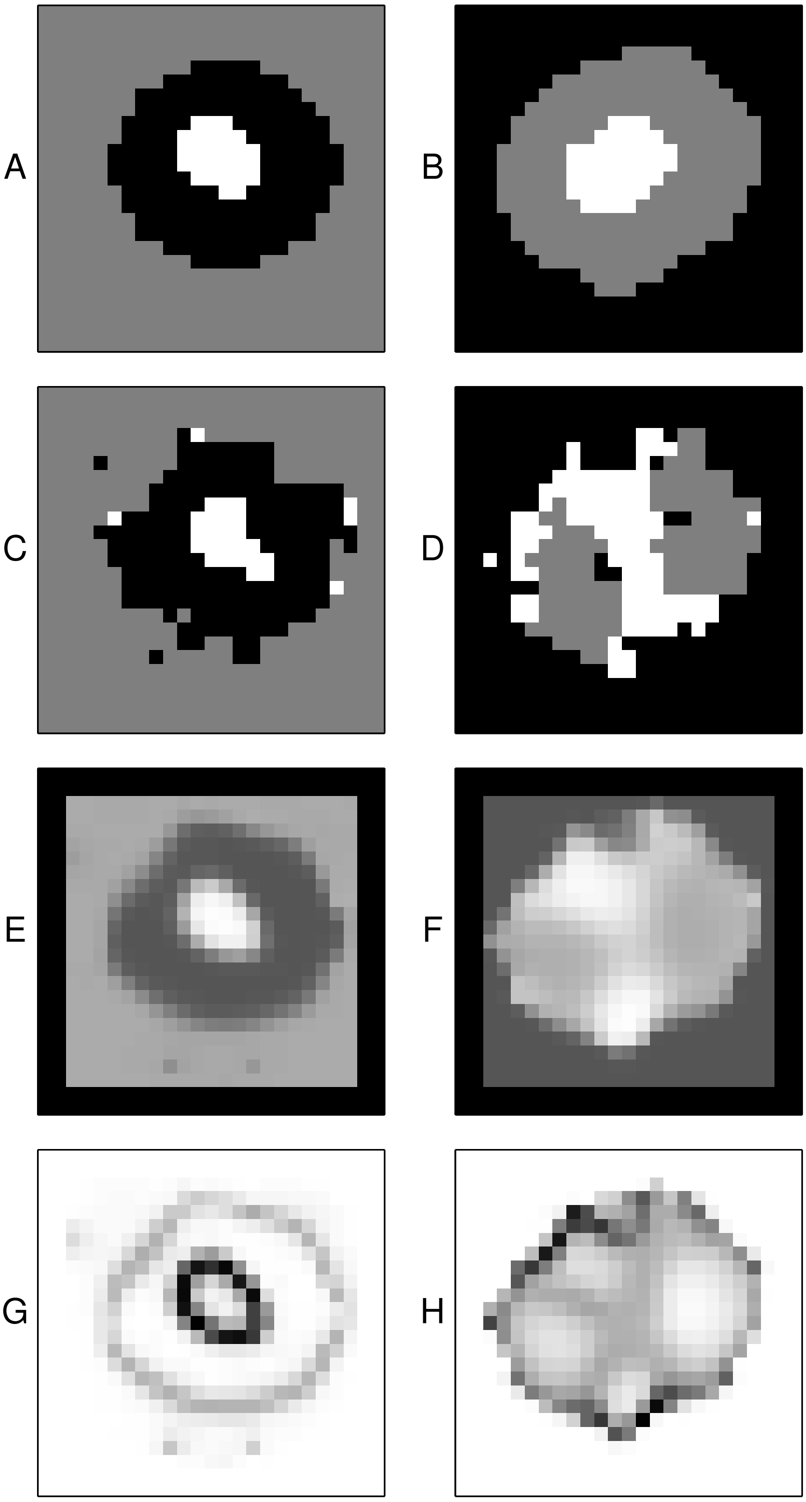
Experiment 2

(continuous variables – three conductivity types)

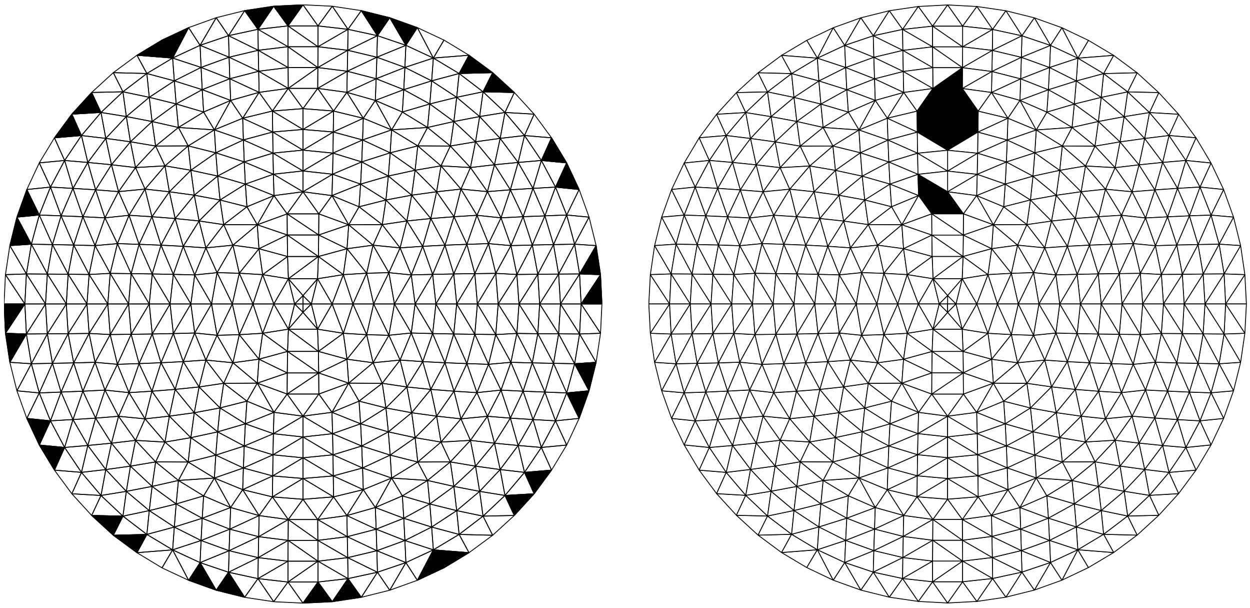


Experiment 3

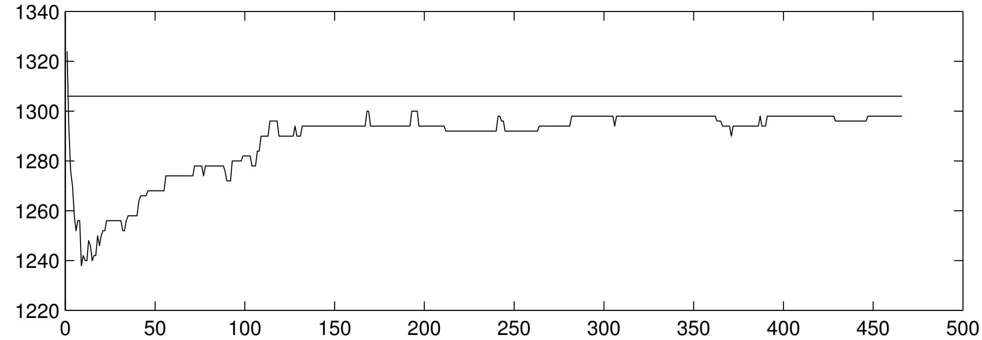
(shielding)

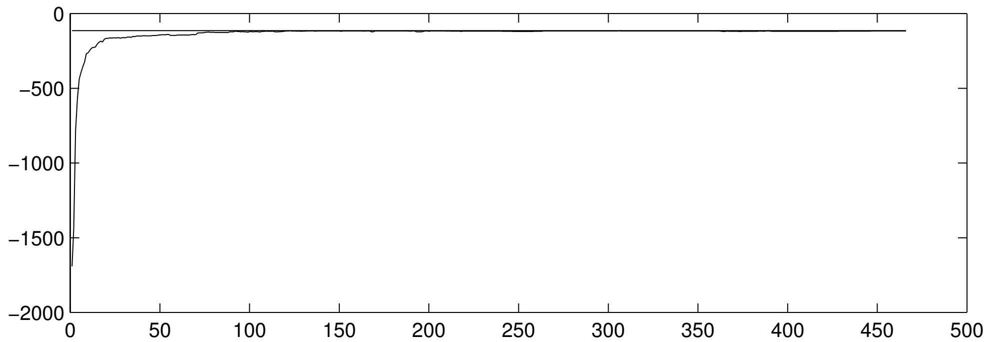


Accurate FEM Model

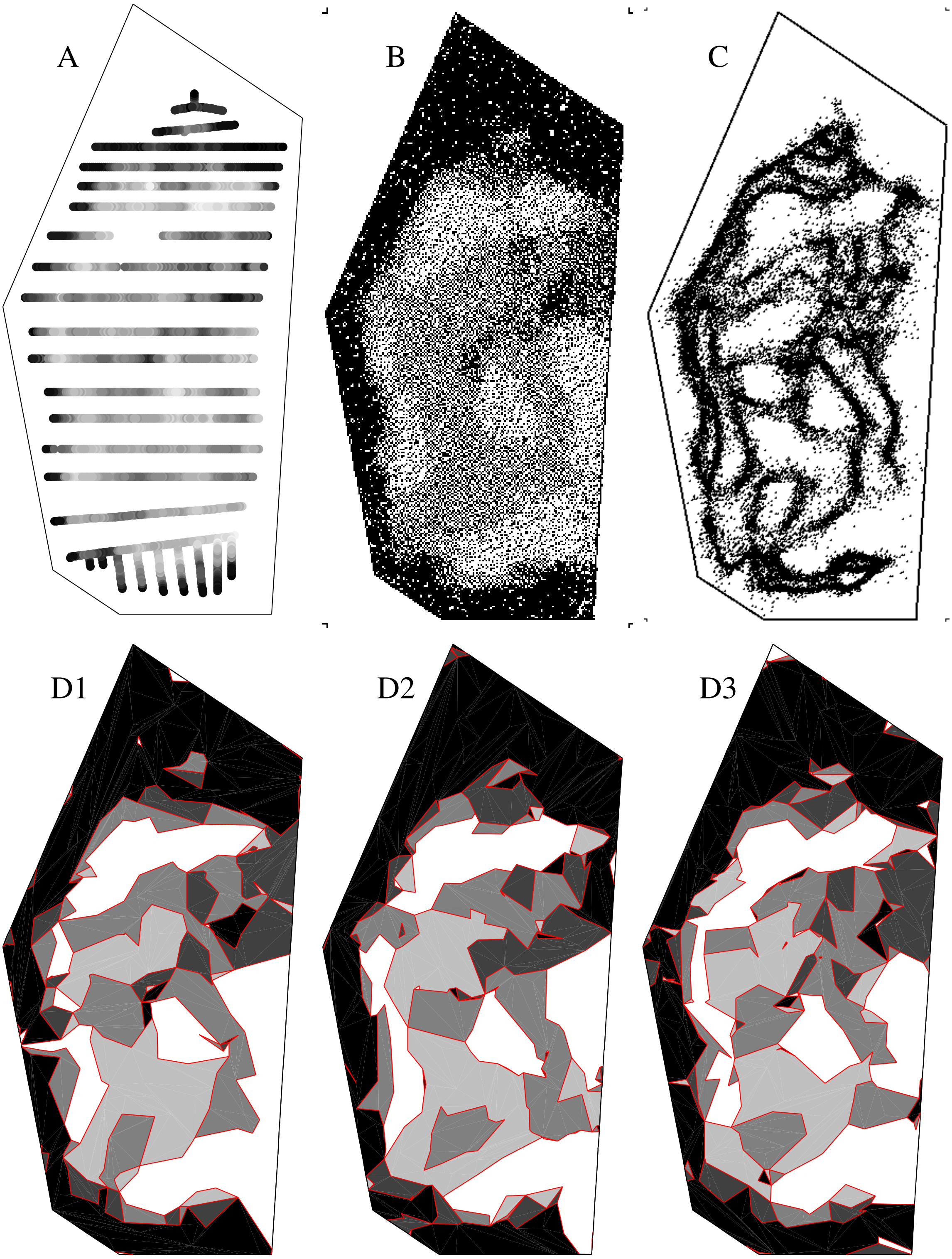


sample 4650

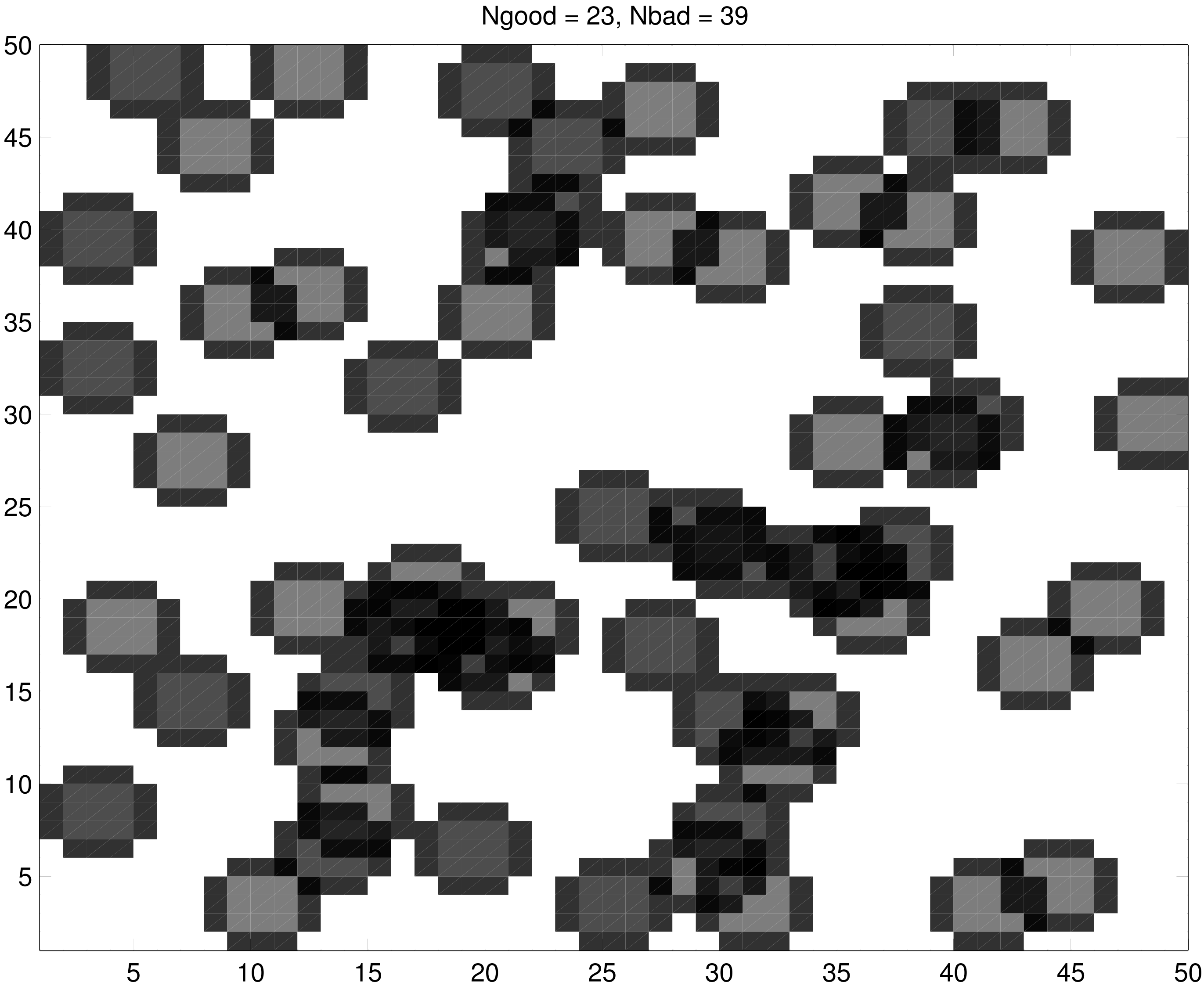




Mid-level Model (triangles)



High-level Model (templates)



Summary

- If you can simulate the forward map then you can sample and calculate expectations over the posterior, i.e., 'solve' the inverse problem
- Statistical inference provides a unifying framework for inverse problems
- Image "analysis" can be part of the "reconstruction"

References

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[2] G. K. Nicholls, "Bayesian image analysis with Markov chain Monte Carlo and coloured continuum triangulation mosaics," *Journal of the Royal Statistical Society B* **60**, pp. 643–659, 1998.

[3] A.J. Baddeley and M.N.M van Lieshout. Stochastic geometry models in high level vision. In KV Mardia and GK Kanji, editors, *Statistics and Images, Vol 1*, volume 20 of *J. Applied Statistics*, pages 231–256. 1993.

[4] G. K. Nicholls and C. Fox, "Prior Modelling and Posterior sampling in Impedance Imaging," In A. Mohammad-Djafari editor, *Bayesian Inference for Inverse Problems*, SPIE conference proceedings volume 3459, pp 116-127, 1998.