A variational approach to modelling the vibration of timber joist floors (and other light timber framed constructions)

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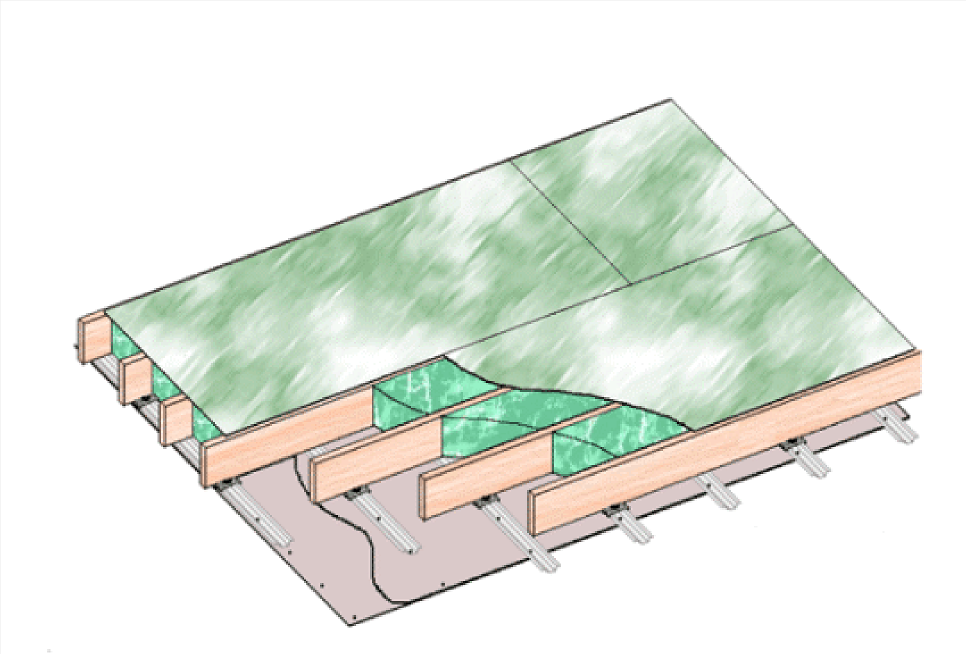
thanks to Grant Emms, George Dodd, Ken McGunnigle

Outline

- What are the issues in modelling sound transmission in timber floors?
- Variational approach
- Fourier basis – (non-local)
- Some calculations and comparison with data model

Typical Timber Floor Construction



- Composite of simple components
- Main elements may be modelled as plates (floor, ceiling) and beams (joists)
- Range of connections possible – nails, screws, glue
- Air cavity conducts sound – may contain dampening material
- Construction is layered and typically rectangular

We would like to calculate:

- Details of structural vibration and hence sound transmission
- from footfall
- particularly for low frequencies (< 200 Hz)
- taking into account construction details (are they important?)

For design purposes we want to know

- Which modes of vibration are important?
- Is slippage at plate/beam joint important?
- Intuitive map (scaling) from material and construction parameters to vibration
- Simple and fast computer code

Lagrangian Variational Formulation

$$\mathcal{L}\left(w(t)\right) = \int_0^T \left[\mathcal{K}\left(t\right) + \mathcal{W}\left(t\right) - \mathcal{P}\left(t\right)\right] dt,$$

$\mathcal{P}$ and $\mathcal{K}$ – potential and kinetic energies

$\mathcal{W}$ – work done on structure by external forces

True motion satisfies

$$\delta\mathcal{L}\left(w(t)\right) = 0$$

Parameterize motion $w(t)$ by 'allowable' modes of vibration

e.g for a single plate

$$\mathcal{P}(t) = \frac{1}{2}\int_V \epsilon_{ij}\tau_{ij} dx\, dy\, dz \qquad K(t) = \frac{1}{2}\int_V \rho\|\dot{w}\|^2 dx\, dy\, dz$$

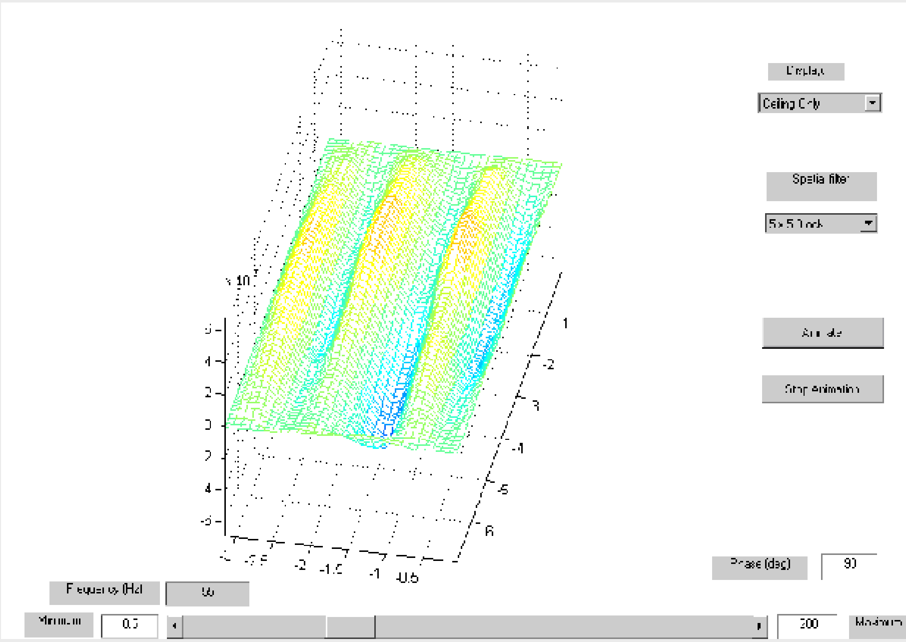
Transverse displacement, lines perpendicular to middle surface S remain so: Kirchhoff model

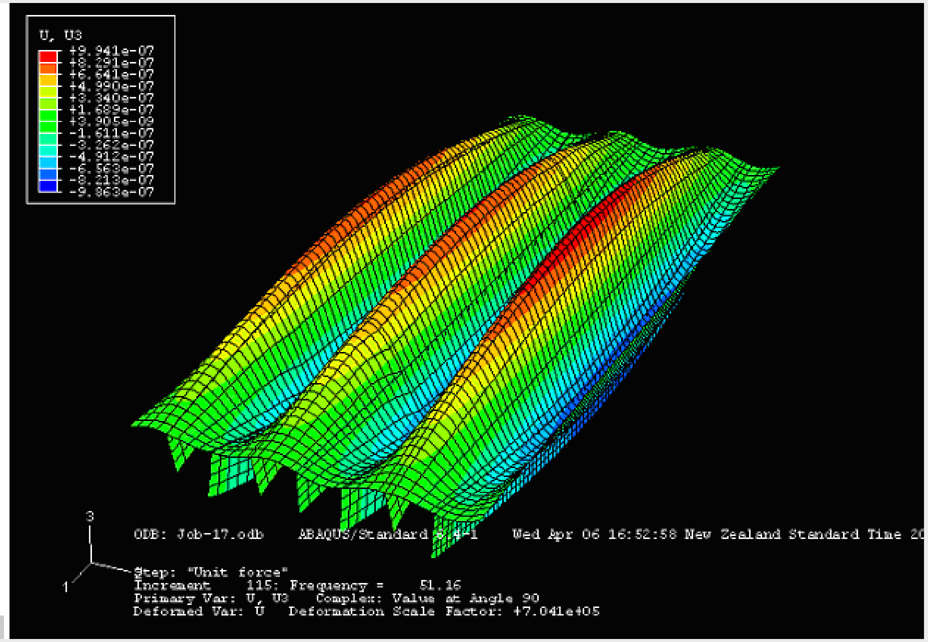
$$\mathcal{P} = \frac{1}{2}\int_S D\left(w_{xx}^2 + w_{yy}^2 + 2\nu w_{xx} w_{yy} + 2\left(1-\nu\right) w_{xy}^2\right)\, dxdy$$

J. E. Lagnese, *Boundary Stabilization of Thin Plates*, SIAM, 1989

Standard Finite Element Method

Triangulation or voxelation of domain, local basis functions, arbitrary node motion





mode (1,6) 55Hz

Fourier Basis Functions

For rectangle $(0, A) \times (0, B)$:

$$w(x, y) = \sum_{m,n=0}^{N} c_{mn} \phi_m(x) \psi_n(y)$$

where

$$\phi_m(x) \propto \exp(k_m x) \quad \psi_n(y) \propto \exp(\kappa_n y) \quad \text{with} \quad k_m = \frac{\pi m}{A} \quad \kappa_n = \frac{\pi n}{B}$$

so that $< \phi_n, \phi_m >= \delta_{nm}$ and $< \psi_n, \psi_m >= \delta_{nm}$

Then

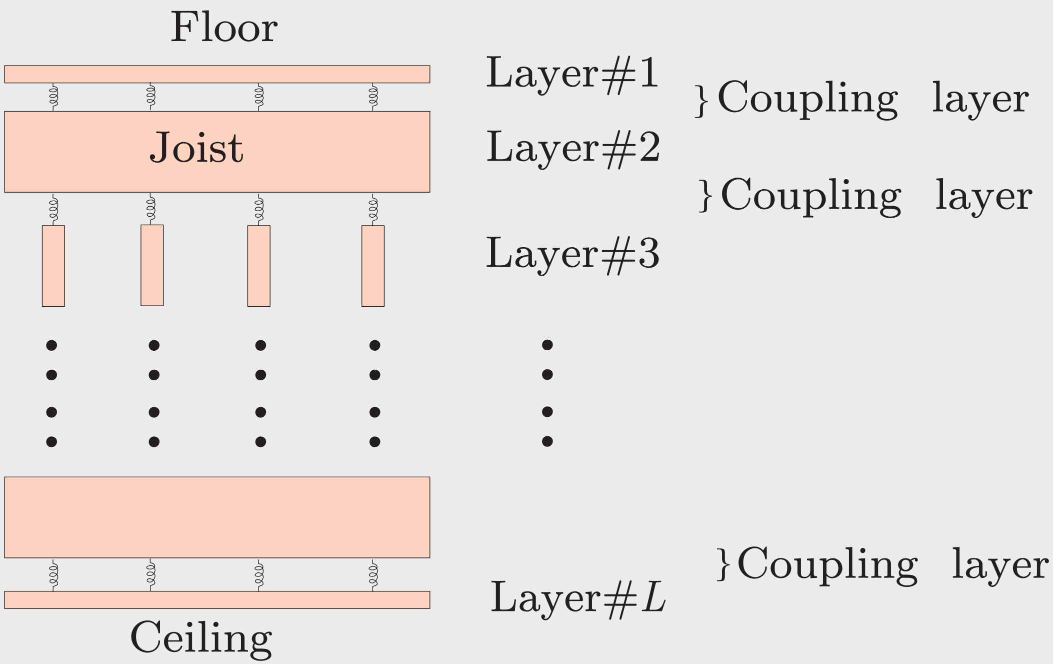
$$\frac{\partial}{\partial x} \equiv k_m \quad \frac{\partial}{\partial y} \equiv \kappa_n$$

Differential terms become algebraic in each component, e.g. Kirchoff plate

$$\mathcal{P} = \frac{1}{2} D c^{\mathsf{T}} \left(k^2 + \kappa^2\right)^2 c$$

where c is vector of coefficients $\{c_{mn}\}$, $k = \text{diag}(k_{mn})$, $\kappa = \text{diag}(\kappa_{mn})$

Elements in a Structure



Motion of structural elements in layer l parameterized by co-efficients

$$c^l = \{c^l_{mn}\}$$

$l = 1, 2, \ldots, L$

Plate layer: $$w^l(x, y) = \sum_{m,n=0}^{N} c^l_{mn} \phi_m(x) \psi_n(y)$$

Joist layer: $$w^l(x, n) = \sum_{m=0}^{N} c^l_{mn} \phi_m(x) \qquad n = 1, 2, \ldots, N^l.$$

$$\mathcal{P} = \frac{1}{2} D c^{l\mathsf{T}} M^l c^l$$

Lagrangian for Elements and Force

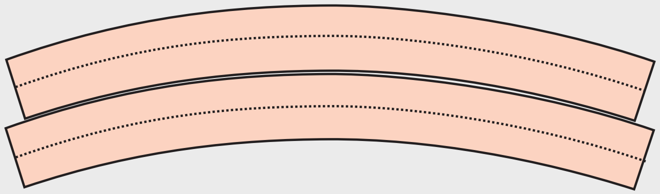
Summing potential energies and work by force on floor

$$\mathcal{P} = \frac{1}{2} D \begin{pmatrix} c^1 \\ c^2 \\ c^3 \\ \vdots \\ c^L \end{pmatrix}^{\mathsf{T}} \begin{bmatrix} M^1 & 0 & 0 & \cdots & 0 \\ 0 & M^2 & 0 & & \\ 0 & 0 & M^3 & & \\ & & & \ddots & \\ & & & & M^L \end{bmatrix} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \\ \vdots \\ c^L \end{pmatrix} + \begin{pmatrix} f^1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} c^1 \\ c^2 \\ c^3 \\ \vdots \\ c^L \end{pmatrix}$$

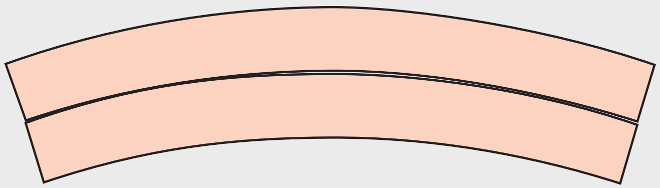
or

$$\mathcal{P} = \frac{1}{2} D c^{\mathsf{T}} M^l c + f^{\mathsf{T}} c$$

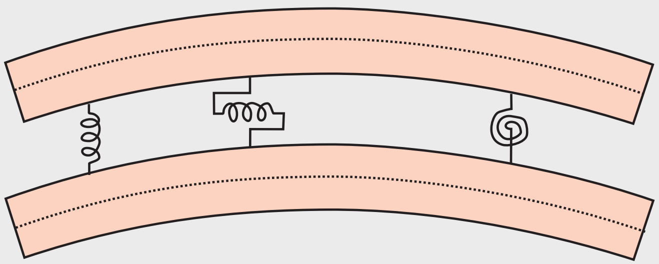
Coupling Layers



Slippage at interface: $2D$



No slippage at interface: $8D$



$$\mathcal{P}_{\mathsf{sep}} = k_{\mathsf{sep}} \left(w^{l} - w^{l+1} \right)^2$$

$$\mathcal{P}_{\mathsf{slip}} = k_{\mathsf{slip}} \left(h^{l} w^{l\prime} + h^{l+1} w^{(l+1)\prime} \right)^2$$

$$\mathcal{P}_{\mathsf{rot}} = k_{\mathsf{rot}} \left(w^{l\prime} - w^{(l+1)\prime} \right)^2$$

Air coupling between plates (floor, ceiling) has same form

Boundary conditions of beams and plates treated analogously

Lagrangian for Elements, Force and Coupling

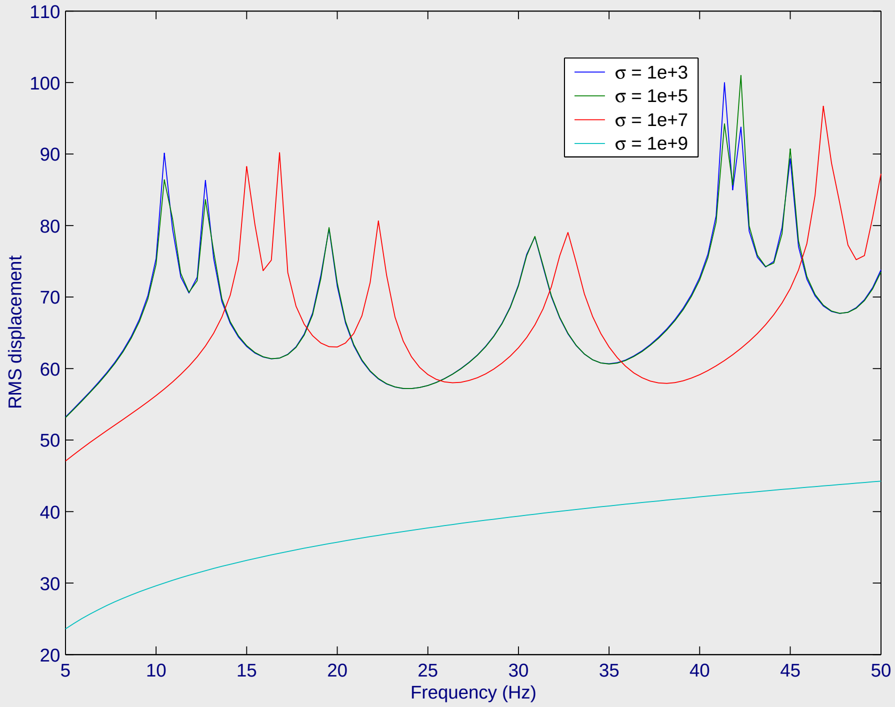
Summing potential energies and work by force on floor

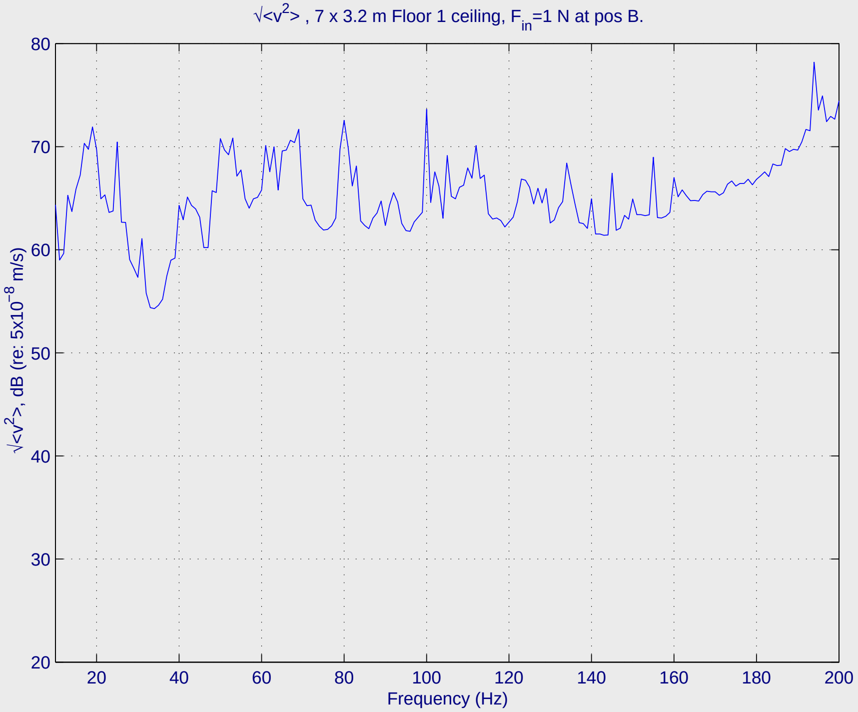
$$\mathcal{P} = \frac{1}{2} D c^{\mathsf{T}} \begin{bmatrix} M^1 + J^{11} & J^{12} & 0 & \cdots & +J^{1L} \\ J^{21} & M^2 + J^{22} & J^{23} & & \\ 0 & J^{32} & M^3 + J^{33} & & \\ & & & \ddots & \\ J^{L1} & & & & M^L + J^{LL} \end{bmatrix} c + f^{\mathsf{T}} c$$

make stationary by solving the normal equations

$$D M_{\mathsf{total}} c = -f$$

Resonances with Slippage at Floor-Joist Joint





Summary

- Computer code is simple and fast (few seconds to run)
- Easy to add new layers and/or coupling conditions
- In-plane (membrane) motion easy to add
- Solutions give intuitively understandable modes of vibration directly
- Flexible formulation fits many constructions